On Newstead's Mayer-Vietoris argument in characteristic 2

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Abstract

Consider the moduli space of framed flat $U(2)$ connections with fixed odd determinant over a surface. Newstead combined some fundamental facts about this moduli space with the Mayer-Vietoris sequence to compute its betti numbers over any field not of characteristic two. We adapt his method in characteristic two to produce conjectural recursive formulae for the mod two betti numbers of the framed moduli space which we partially verify. We also discuss the interplay with the mod two cohomology ring structure of the unframed moduli space.

1 Introduction

Let Σ_g be a compact surface of genus g, and let N_g be the moduli space of flat $SU(2)$ connections on Σ_g having holonomy -1 around a single puncture p. If we write $A_1, B_1, \ldots, A_g, B_g$ for the usual generators of the free group $\pi_1(\Sigma_g \setminus p)$, then N_g is homeomorphic to $f_g^{-1}(-1)/SU(2)$, in which

$$f_g : SU(2)^{2g} \longrightarrow SU(2), \qquad f_g(A_1, B_1, \ldots, A_g, B_g) = \prod_{i=1}^{g} [A_i, B_i],$$

and the action of $SU(2)$, which descends to a free $SO(3)$ action, is by simultaneous conjugation of the $2g$ factors. By a classical result of Narasimhan-Seshadri, N_g may be identified with the moduli space of rank two stable holomorphic bundles over a Riemann surface of genus g with fixed odd determinant. The moduli space of *framed* flat connections is given by

$$N_g^{\#} = f_g^{-1}(-1)$$

and forms an $SO(3)$-principal bundle over the moduli space N_g.

The betti numbers of the moduli space N_g have been computed in a variety of ways. The first way, which was originally done for any coefficient field not of characteristic 2, is due to Newstead [New67]. The argument, which is quite elementary, uses a Mayer-Vietoris sequence to compute formulae for the betti numbers of the framed moduli space which are recursive in g, and then uses the Gysin sequence for the $SO(3)$-fibration $N_g^{\#}$ to obtain the betti numbers for N_g. Subsequently, Harder-Narasimhan [HN75] and Atiyah-Bott [AB83] gave very different and more sophisticated proofs, respectively: the first number-theoretic, and the latter using infinite-dimensional Morse theory on the Yang-Mills functional. These two methods work for higher rank moduli as well. Finally, we mention the elegant proof of Thaddeus [Tha00], which shows that $(A_i, B_i) \longmapsto \mathrm{tr}(A_g)$ is a perfect Morse-Bott function on N_g, as was observed by Jeffrey-Weitsman [JW97].

Newstead's original proof shows that the integral cohomology groups of N_g and $N_g^{\#}$ have no torsion other than 2-torsion. In their work, Atiyah-Bott showed that the integral cohomology of N_g is in fact torsion-free, which can also be seen from the proof of Thaddeus. However, the space $N_g^{\#}$ generally has 2-torsion, as is indicated by the fact that the $g = 1$ framed moduli space, which is a bundle over the point N_1, is homeomorphic to $SO(3)$.

In this article we investigate Newstead's argument in characteristic 2 with the goal of computing the cohomology of $N_g^{\#}$ with $\mathbb{Z}/2$ coefficients. Although we cannot completely compute the betti numbers from the elementary methods used here, we provide evidence for simple recursive formulae similar to Newstead's formulae for the rational betti numbers from [New67]. Specifically, we conjecture that equality holds in all the inequalities appearing in the following:

Theorem 1. *Write $h_r^g = \dim H^r(N_g^{\#};\mathbb{Z}/2)$. Then we have the following:*

$$h_r^{g+1} \geqslant h_{r-2}^g + 2h_{r-3}^g + h_{r-4}^g + m_r^g - m_{r-4}^g \qquad (r \leqslant 3g-1) \qquad (\mathrm{I})_r$$

$$h_r^{g+1} \geqslant 4h_{3g}^g + m_{3g}^g - m_{3g-3}^g \qquad (3g \leqslant r \leqslant 3g+3) \qquad (\mathrm{II})_r$$

$$h_r^{g+1} \geqslant h_{r-2}^g + 2h_{r-3}^g + h_{r-4}^g + m_{r-3}^g - m_{r+1}^g \qquad (r \geqslant 3g+4) \qquad (\mathrm{III})_r$$

in which m_r^g is the coefficient of t^r in the polynomial $(1+t^3)^{2g}$. Further:

(i) Equality holds in $(\mathrm{I})_r$ for $r \equiv 2$ (mod 3) and $r \leqslant 3g-1$.

(ii) Equality holds in the expression for $h_k^{g+1} - h_{k-1}^{g+1}$ obtained by assuming equality in $(\mathrm{I})_r$ for $r \in \{k, k-1\}$ where $k \equiv 1$ (mod 3) and $k \leqslant 3g-1$. Also, $h_{3g+1}^{g+1} = h_{3g}^{g+1}$.

The (in)equalities obtained are immediately doubled: Poincaré duality turns (i) and (ii), which are statements for $r \leqslant 3g+1$, into statements about $r \geqslant 3g+2$. Indeed, $(\mathrm{I})_r$ is transformed into $(\mathrm{III})_r$ via duality, and $(\mathrm{II})_{3g}$ and $(\mathrm{II})_{3g+1}$ into $(\mathrm{II})_{3g+3}$ and $(\mathrm{II})_{3g+2}$, respectively.

The conjectural recursive equations obtained from imposing equality in $(\mathrm{I})_r$–$(\mathrm{III})_r$ are remarkably similar to Newstead's equations for the rational betti numbers of [New67, Thm. 2']: there, equality in $(\mathrm{I})_r$ is satisfied for $r \leqslant 3g+1$, and the rest of the betti numbers follow by Poincaré duality. This small difference in recursions, however, allows the $\mathbb{Z}/2$ betti numbers to grow much larger than the rational ones near the middle dimension. For example, the middle two $\mathbb{Q}$ betti numbers are zero, while our conjecture implies that the four middle $\mathbb{Z}/2$ betti numbers are the same and equal to

$$2^{2g-1} - \binom{2g-1}{g}.$$

The comparison of these betti numbers is further illustrated in Figure 1. The table for the $\mathbb{Z}/2$ betti numbers was computed using Proposition 1 below along with computations from [SS17], and confirms the conjectural recursive formulae for $g \leqslant 6$. Proposition 1 computes the Leray-Serre spectral sequence for the fibration $N_g^{\#} \longrightarrow N_g$ in terms of the rank of multiplication by α on the ring $H^*(N_g;\mathbb{Z}/2)$, where α is the generator of $H^2(N_g;\mathbb{Z}/2)$. We mention that another consequence of the conjecture is the following identity between total ranks:

$\mathbb{Z}/2$ Betti numbers of $N_g^{\#}$

g =	1	2	3	4	5	6
	1	1	1	1	1	1
	1	0	0	0	0	0
		1	1	1	1	1
		5	6	8	10	12
		5	1	1	1	1
			7	8	10	12
			22	29	46	67
			22	9	10	12
				37	46	67
				93	131	232
				93	56	67
					176	233
					386	574
					386	299
						794
						1586
						1586

$\mathbb{Q}$ Betti numbers of $N_g^{\#}$

g =	1	2	3	4	5	6
	1	1	1	1	1	1
	0	0	0	0	0	0
		1	1	1	1	1
		4	6	8	10	12
		0	1	1	1	1
			6	8	10	12
			15	29	46	67
			0	8	10	12
				28	46	67
				56	130	232
				0	45	67
					120	232
					210	561
					0	220
						495
						792
						0

Figure 1: Comparison of the $\mathbb{Z}/2$ and $\mathbb{Q}$ betti numbers of the framed moduli space. The $\mathbb{Z}/p$ betti numbers for p prime, $p \neq 2$ are the same as the $\mathbb{Q}$ betti numbers. In each column half the betti numbers are listed; the rest are obtained by Poincaré duality. For example, the $\mathbb{Z}/2$ betti numbers of $N_2^{\#}$ are 1,0,1,5,5,5,5,1,0,1.

$$\dim_{\mathbb{Z}/2} H^*(N_g^{\#};\mathbb{Z}/2) \;=\; 2\cdot\dim_{\mathbb{Q}} H^*(N_g^{\#};\mathbb{Q}), \tag{1}$$

with the right side known to equal to $2g\binom{2g}{g}$. In fact, the verification of (1) would together with the inequalities of Theorem 1 imply the conjectural recursive equalities.

The proofs of (i) and (ii) and the inequalities in Theorem 1 follow an adaptation of Newstead's Mayer-Vietoris argument. We also provide evidence for a stronger statement than the above conjecture, which may be accessible via geometric methods. The framed moduli space embeds into an extended moduli space N_g^+ which contains the singular locus $f_g^{-1}(+1)$. If it were the case that the maps on homology induced by inclusion, written in the sequel as

$$\nu_r^g : H_r(N_g^{\#};\mathbb{Z}/2) \longrightarrow H_r(N_g^+;\mathbb{Z}/2),$$

were always of *maximal rank*, then our method would carry through to prove that equality holds in Theorem 1. More precisely, we suspect that ν_r^g is surjective for the first half of the $6g-6$ degrees, and injective for the latter half. We will show that ν_r^g is of maximal rank for all r when $g \in \{1,2\}$, although we will only sketch our computations in the $g=2$ case. The manifold N_g^+ may be viewed as a real algebraic deformation of the singular locus $f_g^{-1}(+1)$ with generic fiber homeomorphic to $N_g^{\#}$, and understanding ν_r^g seems an interesting problem in itself.

If the conjectural recursive formulae hold, then $H^*(N_g^{\#};\mathbb{Z})$ is torsion-free in the first 1/3 and last 1/3 of its degrees, and has nontrivial 2-torsion in-between. We can say a bit more about this. It has been mentioned above that our conjectural formulae have been verified for $g \leqslant 6$ using the Leray-Serre spectral sequence and the computations of [SS17]. In that paper, we study the cohomology ring $H^*(N_g;\mathbb{Z}/2)$, and a featured result is that the nilpotency degree of $\alpha \in H^2(N_g;\mathbb{Z}/2)$ is equal to

g. This latter point is related to the current work as follows. Consider the Bockstein homomorphism associated to the short exact coefficient sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$, written

$$\beta : H^r(N_g^\#; \mathbb{Z}/2) \longrightarrow H^{r+1}(N_g^\#; \mathbb{Z}).$$

Using a straightforward induction argument, the conjectural formulae imply that the $\mathbb{Z}/2$ betti numbers and $\mathbb{Q}$ betti numbers of the framed moduli space agree up to degree $r = 2g-2$. Thus we expect that $\beta = 0$ in degrees $r \leqslant 2g-2$. Let $y \in H^1(SO(3); \mathbb{Z}/2)$ be a generator. Then $\alpha^{g-1} \otimes y$ is an element in the E_2 page of the Leray-Serre spectral sequence for the fibration $N_g^\#$. By Proposition 1 below and the nilpotency $\alpha^g = 0$ from [SS17, Thm. 1], it survives to the E_∞ page to define a non-zero element $[\alpha^{g-1} \otimes y] \in H^{2g-1}(N_g^\#; \mathbb{Z}/2)$. This element has no integral lift since y has no integral lift, and thus we obtain the following.

Corollary 1. $\beta\left([\alpha^{g-1} \otimes y]\right) \neq 0$.

From the discussion above, we expect this to account for the first difference between the $\mathbb{Z}/2$ betti numbers and $\mathbb{Q}$ betti numbers, which occurs at $r = 2g-1$. In fact, the conjectural formulae imply that the $\mathbb{Z}/2$ betti number at $r = 2g-1$ is always exactly one more than the $\mathbb{Q}$ betti number, and thus we expect that the element $[\alpha^{g-1} \otimes y]$ entirely accounts for this difference.

Finally, we make a few remarks on other approaches to proving equality in $(\mathrm{I})_r$–$(\mathrm{III})_r$. One might try to apply Thaddeus's Morse-theoretic argument of [Tha00] to the framed moduli space. Indeed, a priori, the function $(A_i, B_i) \longmapsto \mathrm{tr}(A_g)$ defined on $N_g^\#$, the pullback of Thaddeus's function, may be perfect Morse-Bott over $\mathbb{Z}/2$. This is not the case, however: for genus 2, the betti numbers for the starting page of the Bott-Morse spectral sequence with $\mathbb{Z}/2$ coefficients are $1,1,2,6,6,6,6,2,1,1$, while the $\mathbb{Z}/2$ betti numbers of $N_2^\#$, which constitute the E_∞ page, are $1,0,1,5,5,5,5,1,0,1$. The gaps between these pages increases as the genus grows. On a related note, it would be interesting to see if the ∞-dimensional method of Atiyah-Bott [AB83] has anything to say here.

Outline. In Section 2 we fix our notation and record some useful results from [New67]. In Section 3 we compute some data in the genus 1 case in order to apply Newstead's Mayer-Vietoris argument in Section 4 to prove Theorem 1. Finally, in Section 5 we sketch the arguments that show ν_r^g is of maximal rank for genus 2.

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2 Preliminaries

In this section we list some facts from Newstead's paper [New67] and fix notation and conventions. All homology groups will be with $\mathbb{F} = \mathbb{Z}/2$ coefficients unless otherwise indicated, and we write $|V|$ for the dimension of a vector space V. Although we henceforth fix our coefficient field $\mathbb{F}$, it is worth remarking that the results of this section hold for any coefficient field.

Write $SU(2) = D_+ \cup D_-$ as a union of two 3-balls, each with boundary the 2-sphere of trace-free elements, and with $\pm 1 \in D_\pm$. Then define the $6g$-dimensional manifolds with boundary

$$N_g^{\pm} = f_g^{-1}(D_\pm).$$

Newstead explains that N_g^- is homeomorphic to $D_- \times N_g^{\#}$. In particular, the boundaries of both N_g^+ and N_g^- may be identified with $S^2 \times N_g^{\#}$. Define the betti numbers

$$\check{n}_r^g = |H_r(N_g^+)|, \qquad \hat{n}_r^g = |H_r(N_g^+, \partial N_g^+)|.$$

Note that $\check{n}_{6g-r}^g = \hat{n}_r^g$ by Lefschetz duality. Let $\mu_r^g : H_r(\partial N_g^+) \longrightarrow H_r(N_g^+)$ be the map on homology induced by inclusion. Using the Künneth decomposition for the homology of the boundary of N_g^+, we can write μ_r^g as the sum of two maps, ν_r^g and ρ_r^g:

$$\begin{tikzcd}
H_0(S^2) \otimes H_r(N_g^{\#}) \arrow[r, "\nu_r^g"] & H_r(N_g^+) \\
H_2(S^2) \otimes H_{r-2}(N_g^{\#}) \arrow[ur, "\rho_r^g"'] &
\end{tikzcd}$$

Note that the domains of ν_r^g and ρ_r^g are naturally isomorphic to $H_r(N_g^{\#})$ and $H_{r-2}(N_g^{\#})$, respectively. These two maps play a central role in the sequel. Write m_r^g for the betti numbers of $SU(2)^{2g}$. These were given in the introduction as the coefficients of $(1+t^3)^{2g}$. They are explicitly given by:

$$m_r^g = \dim H_r(SU(2)^{2g}) = \begin{cases} \binom{2g}{r/3}, & r \equiv 0 \,(\mathrm{mod}\ 3) \\ 0\,, & \text{otherwise} \end{cases}$$

We now list some elementary relations between the quantities thus far introduced. To start, the following says that the betti numbers of N_g^+ determine those of $N_g^{\#}$ and conversely:

Lemma 1.

$$\check{n}_r^g = \begin{cases} h_{r-2}^g + m_r^g & (r \leqslant 3g+1) \\ h_{r-2}^g - m_{r+1}^g & (r \geqslant 3g+1) \end{cases} \qquad \hat{n}_r^g = \begin{cases} h_{r-1}^g - m_{r-1}^g & (r \leqslant 3g-1) \\ h_{r-1}^g + m_r^g & (r \geqslant 3g-1) \end{cases}$$

This lemma follows from Lemmas 2 and 3 in Section 7 of [New67]. There, Newstead shows that the two maps $H_r(N_g^+) \longrightarrow H_r(SU(2)^{2g})$ and $H_r(N_g^\#) \longrightarrow H_r(SU(2)^{2g})$ induced by inclusion are surjective for $r \leqslant 3g+2$ and $r \leqslant 3g-1$, respectively. His arguments for surjectivity are elementary and easily seen to hold for any coefficient ring. The formula for $\check{n}_r^g$ with $r \leqslant 3g+1$ then follows by looking at the long exact sequence associated to the pair $(SU(2)^{2g}, N_g^+)$ and observing that excision identifies the group $H_r(SU(2)^{2g}, N_g^+)$ with $H_{r-2}(N_g^\#)$. The formula for $\hat{n}_r^g$ with $r \leqslant 3g-1$ follows in a similar way, and the rest of the formulae follow by Lefschetz duality.

Next, we mention that the kernels and cokernels of the maps ρ_r^g and μ_r^g are also determined by the betti numbers of $N_g^\#$. From the long exact sequence of the pair $(N_g^+, \partial N_g^+)$ we have

$$|\mathrm{coker}(\mu_r^g)| + |\ker(\mu_{r-1}^g)| \;=\; |H_r(N_g^+, \partial N_g^+)| \;=\; \hat{n}_r^g,$$

from which the following is easily computed, with help of the above lemma:

$$|\ker(\mu_r^g)| \;=\; \begin{cases} h_r^g - m_r^g & (r < 3g) \\ h_r^g + m_{r+1}^g & (r \geqslant 3g) \end{cases}$$

And for the map ρ_r^g we may consider the Mayer-Vietoris sequence associated to the decomposition of $SU(2)^{2g}$ into the union of N_g^+ and N_g^- along their boundaries:

$$\cdots \;\; H_r(S^2 \times N_g^\#) \xrightarrow{\;\chi_r^g\;} H_r(N_g^+) \oplus H_r(N_g^-) \longrightarrow H_r(SU(2)^{2g}) \;\; \cdots$$

The fact that N_g^- is homeomorphic to $D^3 \times N_g^\#$ implies that the kernel and cokernel of χ_r^g are isomorphic to the kernel and cokernel of ρ_r^g, respectively. From this we have

$$|\mathrm{coker}(\rho_r^g)| + |\ker(\rho_{r-1}^g)| \;=\; |H_r(SU(2)^{2g})| \;=\; m_r^g.$$

Solving for the kernel and cokernel of ρ_r^g amounts to the following very useful observation:

Lemma 2.

1. *If $r \leqslant 3g+1$, then ρ_r^g is injective, and its cokernel has dimension m_r^g. In particular, if also $r \equiv 1, 2 \pmod 3$, then ρ_r^g is an isomorphism.*

2. *If $r \geqslant 3g+1$, then ρ_r^g is surjective, and its kernel has dimension m_{r+1}^g. In particular, if also $r \equiv 0, 1 \pmod 3$, then ρ_r^g is an isomorphism.*

We do not have as easy a way to compute the kernels and cokernels of the maps ν_r^g in general. We will determine these quantities for low genus examples.

3 Getting started with the genus 1 decomposition

Now we begin the adaptation of Newstead's Mayer-Vietoris argument with coefficients in $\mathbb{F}$. It is from this point onwards that the situation differs from the case of a field that has characteristic not equal to 2. We begin by decomposing, as does Newstead, the genus $g+1$ framed moduli space into two parts that are built from genus 1 and genus g data:

$$N^{\#}_{g+1} \; = \; N^{+}_1 \times N^{\#}_g \bigcup_{S^2 \times N^{\#}_1 \times N^{\#}_g} N^{\#}_1 \times N^{+}_g \tag{2}$$

We refer to [New67, §4] for details. Recall here that $N^{\#}_1$ may be identified with $SO(3)$, with betti numbers $1, 1, 1, 1$, and from Lemma 1, those of N^{+}_1 are $1, 0, 1, 3, 1$. We can then fill in most of the data for the maps we considered in the previous section with $g = 1$ in the following table:

r	h^1_r	$\check{n}^1_r$	μ^1_r	ρ^1_r	ν^1_r
0	1	1	1^1_1	0^1_0	1^1_1
1	1	0	0^0_1	0^0_0	0^0_1
2	1	1	1^1_2	1^1_1	$\boxed{1^1_1}$
3	1	3	1^3_2	1^3_1	$\boxed{1^3_1}$
4	0	1	1^1_1	1^1_1	0^1_0
5	0	1	0^0_1	0^0_1	0^0_0

Figure 2: Genus 1 data. The notation a^c_b stands for a linear map $\mathbb{F}^b \longrightarrow \mathbb{F}^c$ of rank a. All entries are computed from the first column from relations in Section 2, except for ν^1_2 and ν^1_3 (boxed) – see Lemma 3.

In fact, all of this data (not including ν^1_2 and ν^1_3) can be deduced from Newstead's table [New67, §5] via universal coefficients. Now consider the Mayer-Vietoris sequence corresponding to (2):

$$\cdots \;\; H_r(S^2 \times N^{\#}_1 \times N^{\#}_g) \xrightarrow{\lambda^{1,g}_r} H_r(N^{+}_1 \times N^{\#}_g) \oplus H_r(N^{\#}_1 \times N^{+}_g) \longrightarrow H_r(N^{\#}_{g+1}) \;\; \cdots$$

Then the exactness of the Mayer-Vietoris sequence yields the following:

$$h^{g+1}_r \; = \; |\mathrm{coker}(\lambda^{1,g}_r)| + |\ker(\lambda^{1,g}_{r-1})| \tag{3}$$

To understand $\lambda^{1,g}_r$ we decompose all of the homology groups using the Künneth Theorem. Before doing this, let us write the two components of $\lambda^{1,g}_r$ as maps in two different directions:

$$H_r(N^{+}_1 \times N^{\#}_g) \longleftarrow H_r(S^2 \times N^{\#}_1 \times N^{\#}_g) \longrightarrow H_r(N^{\#}_1 \times N^{+}_g)$$

Write ι_r^g for the identity map on $H_r(N_g^\#)$. From here we expand the map $\lambda_r^{1,g}$ using the Künneth decompositions of the three homology groups:

$$\begin{tikzcd}
H_0(N_1^+) \otimes H_r(N_g^\#) & H_0(S^2) \otimes H_0(N_1^\#) \otimes H_r(N_g^\#) \arrow[l, "\nu_0^1 \otimes \iota_r^g"'] \arrow[r, "\iota_0^1 \otimes \nu_r^g"] & H_0(N_1^\#) \otimes H_r(N_g^+) \\
H_2(N_1^+) \otimes H_{r-2}(N_g^\#) & H_2(S^2) \otimes H_0(N_1^\#) \otimes H_{r-2}(N_g^\#) \arrow[l, "\rho_2^1 \otimes \iota_{r-2}^g"'] \arrow[ur, "\iota_0^1 \otimes \rho_r^g"'] & \\
 & H_0(S^2) \otimes H_1(N_1^\#) \otimes H_{r-1}(N_g^\#) \arrow[r, "\iota_1^1 \otimes \nu_{r-1}^g"] & H_1(N_1^\#) \otimes H_{r-1}(N_g^+) \\
H_3(N_1^+) \otimes H_{r-3}(N_g^\#) & H_2(S^2) \otimes H_1(N_1^\#) \otimes H_{r-3}(N_g^\#) \arrow[l, "\rho_3^1 \otimes \iota_{r-3}^g"'] \arrow[ur, "\iota_1^1 \otimes \rho_{r-1}^g"'] & \\
 & H_0(S^2) \otimes H_2(N_1^\#) \otimes H_{r-2}(N_g^\#) \arrow[uuul, "\nu_2^1 \otimes \iota_{r-2}^g"] \arrow[r, "\iota_2^1 \otimes \nu_{r-2}^g"] & H_2(N_1^\#) \otimes H_{r-2}(N_g^+) \\
H_4(N_1^+) \otimes H_{r-4}(N_g^\#) & H_2(S^2) \otimes H_2(N_1^\#) \otimes H_{r-4}(N_g^\#) \arrow[l, "\rho_4^1 \otimes \iota_{r-4}^g"'] \arrow[ur, "\iota_2^1 \otimes \rho_{r-2}^g"'] & \\
 & H_0(S^2) \otimes H_3(N_1^\#) \otimes H_{r-3}(N_g^\#) \arrow[uuul, "\nu_3^1 \otimes \iota_{r-3}^g"] \arrow[r, "\iota_3^1 \otimes \nu_{r-3}^g"] & H_3(N_1^\#) \otimes H_{r-3}(N_g^+) \\
 & H_2(S^2) \otimes H_3(N_1^\#) \otimes H_{r-5}(N_g^\#) \arrow[ur, "\iota_3^1 \otimes \rho_{r-3}^g"'] &
\end{tikzcd}$$

Note that each homology group of S^2, $N_1^\#$ and N_1^+ that appears here is isomorphic to $\mathbb{F}$, with the exception of $H_3(N_1^+)$, which is rank 3. In the sequel, it will be convenient to replace each vector space that appears in such a diagram by a dot $\bullet$ as in Figure 5.

Now, if we plug $r = 0, 1, 2$ into this diagram, the kernels and cokernels are easy to compute with what we know thus far; for example, see Figure 4. We obtain the following:

$$|\mathrm{coker}(\lambda_0^{1,g})| = |\mathrm{coker}(\lambda_2^{1,g})| = |\ker(\lambda_2^{1,g})| = 1,$$

$$|\ker(\lambda_0^{1,g})| = |\ker(\lambda_1^{1,g})| = |\mathrm{coker}(\lambda_1^{1,g})| = 0.$$

Using equation (3) we then deduce, for all $g \geqslant 2$, that $h_0^g = 1$, $h_1^g = 0$ and $h_2^g = 1$. The first two of these equalities alternatively follow from Newstead's Theorem 1 [New67], which says that the framed moduli space is simply connected for $g \geqslant 2$.

In trying to compute the kernel of the next map $\lambda_3^{1,g}$ to determine h_3^g, we find that the answer depends on ν_2^1, which we have not yet determined. To help solve for the map ν_2^1 we will look at the genus 2 moduli space. Before proceeding with this, we make a short digression regarding the Leray-Serre spectral sequence for the framed moduli space.

$$
\begin{tikzcd}[row sep=small, column sep=small]
3 & 1 \otimes y^3 \arrow[drr] & 0 & \alpha \otimes y^3 & \langle \psi_i \rangle \otimes y^3 & \delta_2 \otimes y^3 \arrow[drr] & 0 & \alpha\delta_2 \otimes y^3 \\
2 & 1 \otimes y^2 & 0 & \alpha \otimes y^2 & \langle \psi_i \rangle \otimes y^2 & \delta_2 \otimes y^2 & 0 & \alpha\delta_2 \otimes y^2 \\
1 & 1 \otimes y \arrow[drr] & 0 & \alpha \otimes y & \langle \psi_i \rangle \otimes y & \delta_2 \otimes y \arrow[drr] & 0 & \alpha\delta_2 \otimes y \\
0 & 1 \otimes 1 & 0 & \alpha \otimes 1 & \langle \psi_i \rangle \otimes 1 & \delta_2 \otimes 1 & 0 & \alpha\delta_2 \otimes 1 \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}
$$

Figure 3: The E_2-page in the Leray-Serre spectral sequence for $N_2^\#$

The cohomological Leray-Serre spectral sequence for the $SO(3)$-fibration $N_g^\#$ with base space N_g is depicted in Figure 3 for $g = 2$, the details of which will be explained shortly. Write y for the degree 1 generator of $H^*(SO(3))$. Now recall $H_1(N_2) = 0$; in fact, N_g is simply connected [New67, Cor. 1]. Since also $h_1^g = 0$ from above, the d_2 differential on the E_2-page of the spectral sequence must be non-zero on the element $1 \otimes y$. Thus $d_2(1 \otimes y) = \alpha \otimes 1$, and using the Leibniz rule, we obtain that for any $x \in H^*(N_2)$ we have $d_2(x \otimes y^i) = \alpha x \otimes y^{i-1}$ for $i \in \{1, 3\}$, and d_2 is otherwise 0.

From here, the only possible element in E_2 to survive to $H^2(N_2^\#)$ is represented by $1 \otimes y^2$. However, we already computed above that $h_2^2 = 1$, necessitating its survival. Thus the E_i-page differential d_i for $i \geqslant 3$ is zero on the class of $1 \otimes y^2$. Since d_i for $i \geqslant 3$ vanishes on the bottom two rows of the E_i-page for degree reasons, and every element in the top two rows is a multiple of the class of $1 \otimes y^2$, the Leibniz rule implies that d_i vanishes everywhere. Thus we have:

Proposition 1. *For $g \geqslant 2$, the E_2-page differential in the cohomological Leray-Serre spectral sequence for $N_g^\#$ sends $x \otimes y^i$ to the element $\alpha x \otimes y^{i-1}$ for $i \in \{1, 3\}$ and any $x \in H^*(N_g)$, and is otherwise zero. The spectral sequence collapses at the E_3-page. Consequently, we have the formula*

$$
h_r^g \;=\; |\mathrm{coker}(\alpha_{r-2}^g)| + |\ker(\alpha_{r-1}^g)| + |\mathrm{coker}(\alpha_{r-4}^g)| + |\ker(\alpha_{r-3}^g)|
$$

where $\alpha_r^g : H_r(N_g) \longrightarrow H_{r+2}(N_g)$ is the map defined by cup product with α.

Now we explain the genus 2 case more fully. The moduli space N_2 is 6-dimensional, and its cohomology ring over $\mathbb{F}$ is generated be a degree 2 element α, degree 3 elements $\psi_1, \psi_2, \psi_3, \psi_4$, and a degree 4 element δ_2. The ring structure is determined by the following: the only top degree monomials that pair nontrivially with the fundamental class $[N_2]$ are the following:

$$
\alpha\delta_2, \quad \psi_1\psi_3, \quad \psi_2\psi_4.
$$

In particular, $\alpha^2 = 0$. This ring, and in fact the corresponding ring with integer coefficients, is described in Remark 2 of Section 10 in [New67].

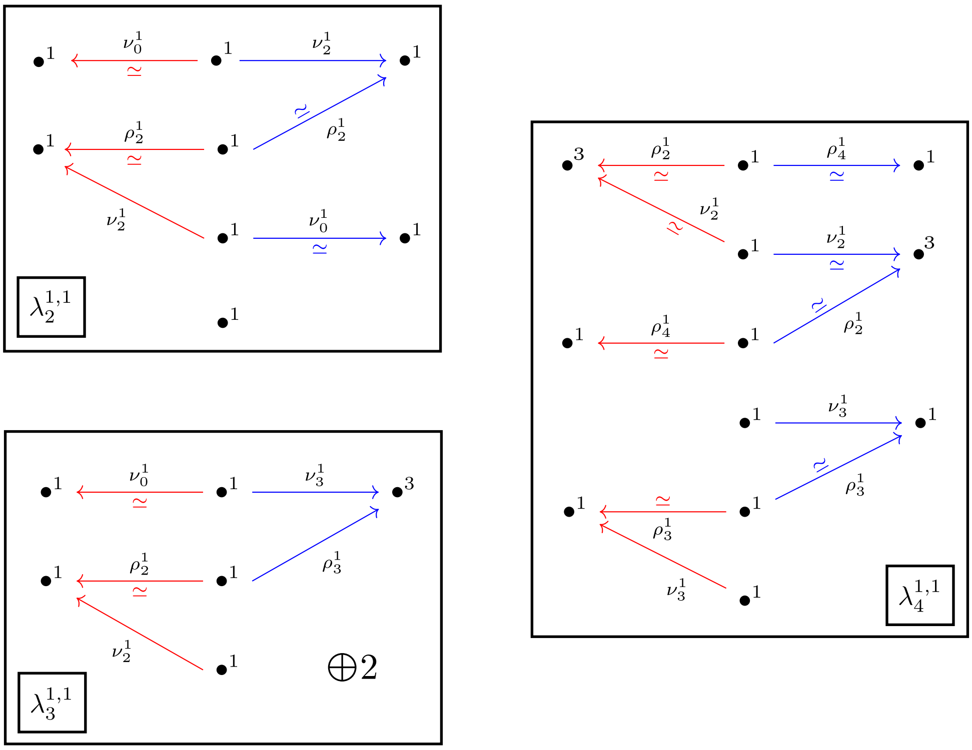

Figure 4: The maps $\lambda_2^{1,1}$, $\lambda_3^{1,1}$ and $\lambda_4^{1,1}$. Each vector space has been replaced by a dot $\bullet$. The dimension of each vector space is written as a superscript of each $\bullet$. The notation $\oplus 2$ in the lower left pane indicates that the map $\lambda_3^{1,1}$ consists of two copies of the depicted map. The lone lower dot in the upper left pane of $\lambda_2^{1,1}$ comes from the domain of $\iota_1^1 \otimes \nu_1^1$.

Now Figure 3 is obtained from this description of the ring and Proposition 1. The arrows drawn represent the non-trivial E_2 differentials. Note that we have written $\langle \psi_i \rangle$ for the 4-dimensional vector space with basis the ψ_i classes. The numbers h_r^2 are then computed from Figure 3 to be

$$1,\ 0,\ 1,\ 5,\ 5,\ 5,\ 5,\ 1,\ 0,\ 1. \tag{4}$$

We are now in a position to compute ν_2^1. Consider the map $\lambda_3^{1,1}$. Referring to Figure 4, we find that the cokernel of this map is 4 or 6, depending on whether ν_2^1 is an isomorphism or not, respectively. Since we now know that $h_3^2 = 5$, and from above $|\ker(\lambda_2^{1,1})| = 1$, equation (3) implies that ν_2^1 must in fact be an isomorphism. We now have the first part of:

Lemma 3. *The maps ν_2^1 and ν_3^1 are injective.*

To compute ν_3^1 we next consider $\lambda_4^{1,1}$. Referring again to Figure 4, we find that the cokernel of $\lambda_4^{1,1}$ has dimension equal to 5 or 6 depending on whether ν_3^1 is injective or not, respectively. We are using our knowledge that the image of ν_3^1 is contained in that of ρ_3^1, as follows from μ_3^1 having rank 1. From above, the kernel of $\lambda_3^{1,1}$ has dimension 0. Finally, from (4) we have $h_4^2 = 5$, and this forces

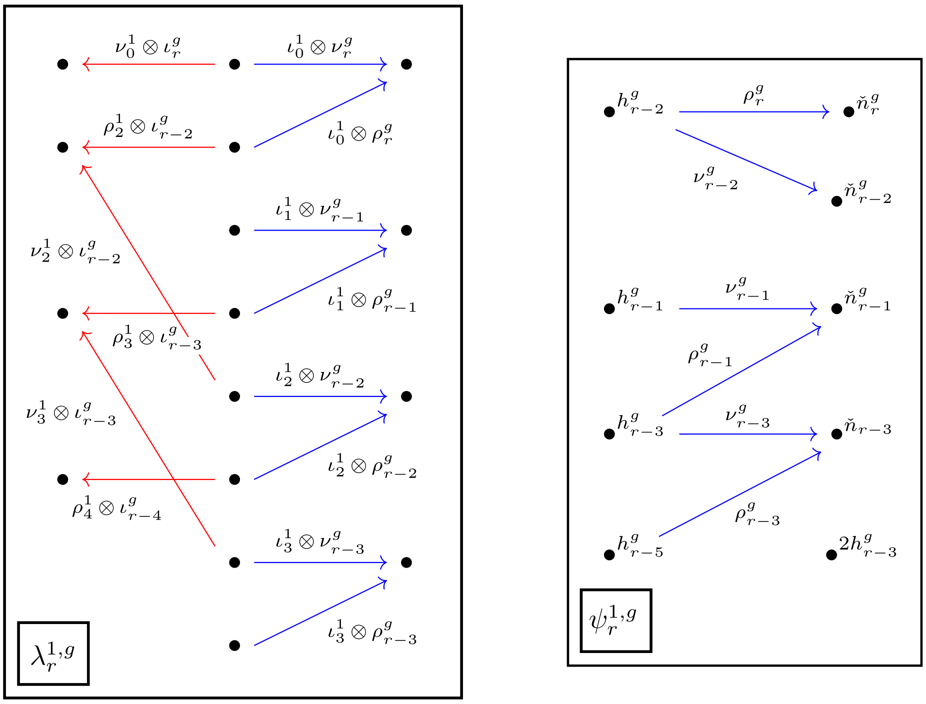

Figure 5: In the left hand pane, we have simply redrawn the above expansion of $\lambda_r^{1,g}$ with a dot $\bullet$ replacing the name of each vector space. The computation of all the left hand (red) maps in this pane allows us to replace $\lambda_r^{1,g}$ with the map $\psi_r^{1,g}$ defined in the right hand pane.

via (3) the dimension of the cokernel of $\lambda_4^{1,1}$ to be 5, implying that ν_3^1 is injective. This completes the proof of the lemma.

4 Applying the Mayer-Vietoris argument

With all of the genus 1 data computed, we are now in a position to prove Theorem 1. Referring to Figure 5, we first replace $\lambda_r^{1,g}$ with a map $\psi_r^{1,g}$ that has the same kernel and cokernel. We will shortly focus on this latter map.

Going from $\lambda_r^{1,g}$ to its simplification $\psi_r^{1,g}$ is only a matter of linear algebra over $\mathbb{F}$. In fact, from the diagrammatic perspective, it is a standard manipulation in the context of computing homology groups over $\mathbb{F}$, usually referred to there as Gaussian elimination. For example, when an arrow is an isomorphism and no other arrow touches its codomain, then we can eliminate the arrow, along with its domain and codomain. This rule allows us to erase from $\lambda_r^{1,g}$ the top left arrow $\nu_0^1 \otimes \iota_r^g$ as well as the arrow corresponding to $\rho_4^1 \otimes \iota_{r-4}^g$. Next, the fact that ν_2^1 and ρ_2^1 are isomorphisms allows us to join the domains of $\iota_0^1 \otimes \rho_r^g$ and $\iota_2^1 \otimes \nu_{r-2}^g$. We can do the same for ν_3^1 and ρ_3^1 to join the domains of $\iota_1^1 \otimes \rho_{r-1}^g$ and $\iota_3^1 \otimes \nu_{r-3}^g$, except that ν_3^1 and ρ_3^1 are only isomorphisms onto their common images: we must also save a complement of this image in their codomain, which will be of dimension $2h_{r-1}^g$. The result after doing these manipulations is the diagram defining $\psi_r^{1,g}$.

Now Lemma 2 allows us to compute the kernel and cokernel of $\psi_r^{1,g}$ in many cases. For example, suppose that $r \leqslant 3g+1$ and $r \equiv 1, 2 \pmod 3$. Then the part of the map consisting of ρ_r^g and ν_{r-2}^g in the diagram for $\psi_r^{1,g}$ does not contribute to the kernel, since ρ_r^g is injective. Also, ρ_r^g is an isomorphism, so it along with its domain and codomain can be eliminated from consideration. After this, as far as the kernel goes, we are left only with the part of the map consisting of ν^g_{r-1} and ρ^g_{r-1}, which is exactly μ^g_{r-1}. For $r \leqslant 3g$ we have $|\ker(\mu^g_{r-1})| = h^g_{r-1} - m^g_{r-1}$, while for $r = 3g+1$ we have instead $|\ker(\mu^g_{3g})| = h^g_{3g}$. We have deduced the first two parts of:

Lemma 4.

1. *If* $r < 3g+1$ *and* $r \equiv 1, 2 \pmod 3$, *then* $|\ker(\lambda^{1,g}_r)| = h^g_{r-1} - m^g_{r-1}$.
2. *If* $r = 3g+1$ *then* $|\ker(\lambda^{1,g}_{3g+1})| = h^g_{3g}$.
3. *If* $r \geqslant 3g+4$ *and* $r \equiv 0, 1 \pmod 3$, *then* $|\ker(\lambda^{1,g}_r)| = h^g_{r-1} + m^g_r$.
4. *If* $r < 3g+1$ *and* $r \equiv 1, 2 \pmod 3$, *then* $|\mathrm{coker}(\lambda^{1,g}_r)| = 2h^g_{r-3} + h^g_{r-4} + m^g_{r-2}$.
5. *If* $r = 3g+1$ *then* $|\mathrm{coker}(\lambda^{1,g}_{3g+1})| = 2h^g_{3g-2} + h^g_{3g-3} + m^g_{3g}$.
6. *If* $r \geqslant 3g+4$ *and* $r \equiv 0, 1 \pmod 3$, *then* $|\mathrm{coker}(\lambda^{1,g}_r)| = 2h^g_{r-3} + h^g_{r-4} - m^g_{r-1}$.

The third item in the lemma is proven similarly: in this range, ρ^g_r is an isomorphism, so again the top part of the diagram for $\psi^{1,g}_r$ contributes no kernel. The map ρ^g_{r-3} is also an isomorphism, and in the same way as before we identify the kernel of $\psi^{1,g}_r$ with that of μ^g_{r-1}. The only difference is that in this range we have $|\ker(\mu^g_{r-1})| = h^g_{r-1} + m^g_r$. The latter three items of the lemma follow from the first three by simply inspecting the dimensions of the domain and codomain of $\lambda^{1,g}_r$.

Proof of (i)-(ii) in Thm. 1. Substitute the items of Lemma 4 into (3). □

Lemma 5. *For all r, the inequalities* $(\mathrm{I})_r$–$(\mathrm{III})_r$ *are valid. Equality holds if and only if for all r,*

$$|\ker(\rho^g_r) \cap \ker(\nu^g_{r-2})| \;=\; 0. \tag{5}$$

Proof. We first note that (5) holds whenever ρ^g_r is injective. Thus Lemma 2 implies (5) for the ranges $r \geqslant 3g+1$ with $r \equiv 0, 1 \pmod 3$, and $r \leqslant 3g+1$. We now focus on the cases in which $r \geqslant 3g+2$ and $r \equiv 2 \pmod 3$. First suppose $r \geqslant 3g+4$ and $r \equiv 2 \pmod 3$. Referring to the diagram for $\psi^{1,g}_r$ in Figure 5, and using the fact that ρ^g_{r-3} is surjective, the kernel is seen to have dimension

$$|\ker(\lambda^{1,g}_r)| \;=\; |\ker(\rho^g_{r-3})| + |\ker(\mu^g_{r-1})| + |\ker(\rho^g_r) \cap \ker(\nu^g_{r-2})|. \tag{6}$$

Using our formulae from Section 2, this kernel is equal to $m^g_{r-2} + h^g_{r-1}$ if and only if (5) holds. In case $|\ker(\rho^g_r) \cap \ker(\nu^g_{r-2})| = 0$ does hold, the cokernel is given by

$$|\mathrm{coker}(\lambda^{1,g}_r)| \;=\; 2h^g_{r-3} + h^g_{r-4} - m^g_{r+1}.$$

Together with items 3 and 6 from Lemma 4 we derive $(\mathrm{III})_r$ for $r \geqslant 3g+5$ with equality holding, and by Poincaré duality, $(\mathrm{I})_r$ for $r \leqslant 3g-2$ with equality. The case of $r = 3g+2$ is similarly handled. From this argument it is clear that (5) holds if and only if $(\mathrm{I})_r$–$(\mathrm{III})_r$ are equalities, and that more generally $|\ker(\rho^g_r) \cap \ker(\nu^g_{r-2})| \geqslant 0$ implies the inequalities $(\mathrm{I})_r$–$(\mathrm{III})_r$ for all r. □

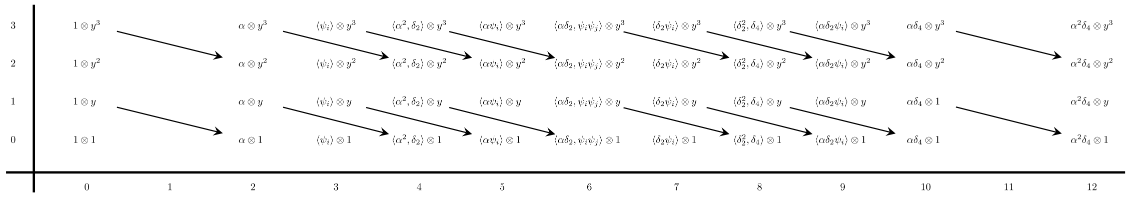

Figure 6: The E_2 page in the Leray-Serre spectral sequence for $N_3^\#$.

Recall from the introduction our claim that equality in $(\mathrm{I})_r$–$(\mathrm{III})_r$ follows if ν_r^g has maximal rank for all r. We explain this here for $(\mathrm{I})_r$. For the range beyond the middle dimension, this asks for ν_r^g to be injective, and thus our claim from the introduction follows from Lemma 5. However, we can also see how surjectivity of ν_r^g in the range below the middle dimension would suffice: here the kernel of $\psi_r^{1,g}$ is computed to have dimension $h_{r-1}^g - m_{r-1}^g - m_{r-3}^g$. This is obtained by splitting off the kernel of ν_{r-1}^g, which contributes $h_{r-1}^g - \check{n}_{r-1}^g$, then cancelling the remaining isomorphic part of ν_{r-1}^g against ρ_{r-1}^g, and accounting for the kernel of $\mu_{r-3}^g = \nu_{r-3}^g \oplus \rho_{r-3}^g$ left over. Computing the cokernels and applying (3) yields equality in $(\mathrm{I})_r$.

5 Computations for the genus 2 decomposition

In this final section we sketch the computations that show ν_r^g is of maximal rank for all r with $g = 2$. None of these are needed for the results stated in the introduction.

One might try to prove equality in $(\mathrm{I})_r$–$(\mathrm{III})_r$ by using other Mayer-Vietoris decompositions. For example, moving a level down from (2), we may consider the genus 2 decomposition

$$N_{g+1}^\# \;=\; N_2^+ \times N_{g-1}^\# \bigcup_{S^2 \times N_2^\# \times N_{g-1}^\#} N_2^\# \times N_{g-1}^+ \tag{7}$$

which may be described in a similar manner as was the genus 1 decomposition (2) in [New67, §4]. Just as in the previous case, we consider the Mayer-Vietoris sequence associated with (7). We have a map $\lambda_r^{2,g-1}$ which we decompose into two parts, as follows:

$$H_r(N_2^+ \times N_{g-1}^\#) \longleftarrow H_r(S^2 \times N_2^\# \times N_{g-1}^\#) \longrightarrow H_r(N_2^\# \times N_{g-1}^+)$$

We also have the analogue of (3) from the exactness of the Mayer-Vietoris sequence:

$$h_r^{g+1} \;=\; |\mathrm{coker}(\lambda_r^{2,g-1})| + |\mathrm{ker}(\lambda_{r-1}^{2,g-1})| \tag{8}$$

As before, we expand $\lambda_r^{2,g-1}$ into its various Künneth components, and obtain the diagram in Figure 7. Here we note that the betti numbers $\check{n}_r^{g-1}$ of N_{g-1}^+ are easily computed from our knowledge of h_r^2 from Section 3 and the equations in Section 2. These are listed in Figure 8. All of the unboxed data in the table is computed from the formulae in Section 2. We will momentarily sketch how one can fill in the boxed data. Here we remark that after computing this data and attempting to adapt the Mayer-Vietoris argument of Section 4 to this situation, it becomes apparent that more information

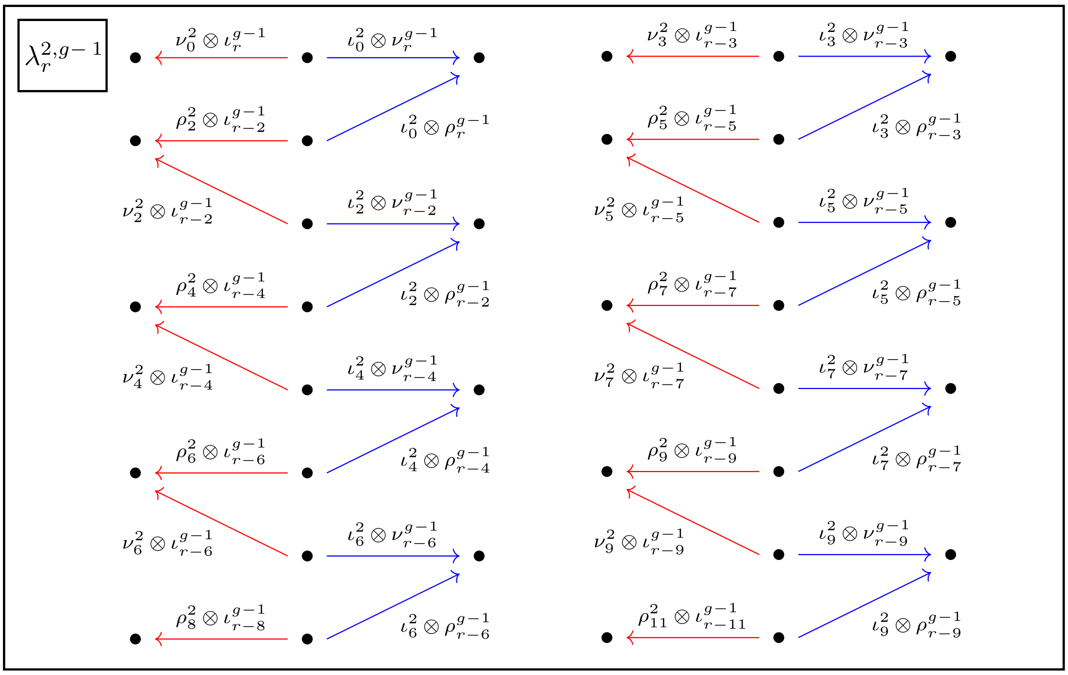

Figure 7: The map $\lambda_r^{2,g-1}$ expanded using the Künneth Theorem.

about the maps ν_r^g and their interactions with the ρ_r^g is required in order to compute the relevant kernels and cokernels.

We can compute the data in Figure 8 by specializing to the $2+1$ and $2+2$ Mayer-Vietoris decompositions, setting $g=2$ and $g=3$ in (7). To carry this out we need the $\mathbb{Z}/2$ betti numbers of the moduli spaces $N_3^\#$ and $N_4^\#$. These are computed via Proposition 1, which uses the Leray-Serre spectral sequence, and the ring structures of $H^*(N_g;\mathbb{F})$ for $g\in\{3,4\}$, which are available from [SS17]. See Figure 3 for an illustration of the genus 3 case. The numbers obtained are of course what appear in Figure 1, and agree with the general conjectural recursions.

The $2+1$ Mayer-Vietoris decomposition of the genus 3 moduli space, which can also be viewed as one of the genus 1 decompositions (2), can be used to compute the following, in the listed order:

1. Use $\lambda_3^{1,2}$ to conclude that ν_2^2 is an isomorphism.

2. Use $\lambda_{11}^{1,2}$ to conclude that ν_9^2 is an isomophism.

3. Use $\lambda_6^{1,2}$ to conclude that $|\ker(\nu_3^2)\cap\ker(\mu_5^2)|=1$, implying that ν_5^2 has rank 4 or 5.

4. Use $\lambda_8^{1,2}$ to conclude that $|\ker(\nu_6^2)\cap\ker(\rho_8^1)|=0$, implying that ν_6^2 has rank 4 or 5.

In each step we use the diagram of maps in Figure 7 with $g=2$, the appropriate value of r, and linear algebra over $\mathbb{F}$ just as in Section 4. In particular, the key device is our use of the relation (8) along with our aforementioned knowledge of the betti numbers h_r^3, which constrains the possible dimensions of the kernels and cokernels of $\lambda_r^{1,2}$. We mention that we can deduce a bit more than what is listed in item 3, from its computation: the kernel of the map $\nu_3^2(\rho_5^2)^{-1}\nu_5^2$ is 1-dimensional. This information is useful for the reader who wishes to complete the subsequent steps.

r	h^2_r	$\check{n}^2_r$	μ^2_r	ρ^2_r	ν^2_r
0	1	1	1^1_1	0^1_0	1^1_1
1	0	0	0^0_0	0^0_0	0^0_0
2	1	1	1^1_2	1^1_1	$\boxed{1^1_1}$
3	5	4	4^4_5	0^4_0	4^4_5
4	5	1	1^1_6	1^1_1	$\boxed{1^1_5}$
5	5	5	5^5_{10}	5^5_5	$\boxed{5^5_5}$
6	5	11	5^{11}_{10}	5^{11}_5	$\boxed{5^{11}_5}$
7	1	5	5^5_6	5^5_5	$\boxed{1^5_1}$
8	0	1	1^1_5	1^1_5	0^1_0
9	1	1	1^1_2	1^1_1	$\boxed{1^1_1}$
10	0	0	0^0_0	0^0_0	0^0_0
11	0	0	0^0_1	0^0_1	0^0_0

Figure 8: Genus 2 data.

We may then proceed to use the $2+2$ decomposition of the genus 4 moduli space in a similar fashion to complete the following two steps:

5. Use $\lambda^{2,2}_7$ to conclude that ν^2_4 is non-zero, and hence surjective.
6. Use $\lambda^{2,2}_{13}$ to conclude that ν^2_7 is non-zero, and hence injective.

It then remains to show that the ranks of ν^2_5 and ν^2_6 are 5, instead of 4. This computation is less direct. However, the joint constraints imposed by inspecting $\lambda^{2,2}_r$ for $r = 8, 9, 10, 12$ lead to the resolution of this claim, which, although entirely elementary, is somewhat tedious. For the reader interested in following this computation through we include the following table, which lists the final dimensions for some of the relevant kernels and cokernels.

r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
h^4_r	1	0	1	8	1	8	29	9	37	93	93	93	93	37	9	29	8	1	8	1	0	1
$\lvert\mathrm{cok}(\lambda^{2,2}_r)\rvert$	1	0	1	8	1	8	29	8	29	68	85	68	85	20	1	12	0	0	0	0	0	0
$\lvert\ker(\lambda^{2,2}_r)\rvert$	0	0	0	0	0	0	1	8	25	8	25	8	17	8	17	8	1	8	1	0	1	0

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