Journal of Pure and Applied Algebra 219 (2015) 2046–2061

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Infinitesimal Hecke algebras of $\mathfrak{so}_N$

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ARTICLE INFO

Article history:
Received 17 February 2014
Received in revised form 10 July 2014
Available online 12 August 2014
Communicated by S. Donkin

ABSTRACT

In this article we classify all infinitesimal Hecke algebras of $\mathfrak{g} = \mathfrak{so}_N$. We establish isomorphism of their universal versions and the W-algebras of $\mathfrak{so}_{N+2m+1}$ with a 1-block nilpotent element of the Jordan type $(1, \ldots, 1, 2m + 1)$. This should be considered as a continuation of [9], where the analogous results were obtained for the cases of $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{sp}_{2n}$.

0. Introduction

In this paper we consider infinitesimal Hecke algebras of $\mathfrak{so}_N$.[1] Although their theory runs along similar lines as for the cases of $\mathfrak{gl}_N$ and $\mathfrak{sp}_{2N}$, they have not been investigated before.

We obtain the classification result in Theorem 1.4 (compare to [5, Theorem 4.2]), compute the Poisson center of the corresponding Poisson algebras in Theorem 4.2 (compare to [4, Theorems 5.1 and 7.1]), compute the first non-trivial central element in Theorem 6.1 (compare to [4, Theorem 3.1]) and derive the isomorphism with the corresponding W-algebras in Theorems 5.1, 5.2 (compare to [9, Theorems 7 and 10]).

Together with [9], this covers all basic cases of the infinitesimal Hecke algebras on the one side and the classical W-algebras with a 1-block nilpotent element, on the other. However, we would like to emphasize that the theory of infinitesimal/continuous Hecke algebras is much more complicated in general and has not been developed yet.

This paper is organized as follows:

- In Section 1, we recall the definitions of the continuous and infinitesimal Hecke algebras of type (G, V) (respectively $(\mathfrak{g}, V)$). We formulate Theorems 1.3 and 1.4, which classify all such algebras for the cases of (SO_N, V_N) and $(\mathfrak{so}_N, V_N)$, respectively.

  We also recall the definitions and basic results about the finite W-algebras.
- In Section 2, we prove Theorem 1.3.
- In Section 3, we prove Theorem 1.4 by computing explicitly the corresponding integral.

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[1] We assume that $N \geq 3$.

http://dx.doi.org/10.1016/j.jpaa.2014.07.022
0022-4049/

- In Section 4, we compute the Poisson center of the classical analogue $H_\zeta^{\mathrm{cl}}(\mathfrak{so}_N, V_N)$.
- In Section 5, we introduce the universal length m infinitesimal Hecke algebra $H_m(\mathfrak{so}_N, V_N)$. In Theorem 5.1 (and its Poisson counterpart Theorem 5.2) we establish an abstract isomorphism between the algebras $H_m(\mathfrak{so}_N, V_N)$ and the W-algebras $U(\mathfrak{so}_{N+2m+1}, e_m)$.
- In Section 6, we find a non-trivial central element of $H_\zeta(\mathfrak{so}_N, V_N)$, called the *Casimir element* of $H_\zeta(\mathfrak{so}_N, V_N)$. This can be used to establish the isomorphism of Theorem 5.1 explicitly.

1. Basic definitions

1.1. Algebraic distributions

For an affine scheme X of finite type over $\mathbb{C}$, let $\mathcal{O}(X)$ be the algebra of regular functions on X and $\mathcal{O}(X)^*$ be the dual space, called the space of *algebraic distributions*. Note that $\mathcal{O}(X)^*$ is a module over $\mathcal{O}(X)$: for $f \in \mathcal{O}(X)$, $\mu \in \mathcal{O}(X)^*$ we can define $f \cdot \mu$ by $\langle f \cdot \mu, g\rangle = \langle \mu, fg\rangle$ for all $g \in \mathcal{O}(X)$. For a closed subscheme $Z \subset X$, we say that an algebraic distribution μ on X is supported on the scheme Z if μ annihilates the defining ideal $I(Z)$ of Z. If Z is reduced, we say that $\mu \in \mathcal{O}(X)^*$ is *set-theoretically supported* on the set Z if μ annihilates some power of $I(Z)$.

Let G be a reductive algebraic group and $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional algebraic representation of G. First note that $\mathcal{O}(G)^*$ is an algebra with respect to the convolution. Moreover, δ_{1_G} is the unit of this algebra. Next, we consider the semi-direct product $\mathcal{O}(G)^* \ltimes TV$, that is, the algebra generated by $\mu \in \mathcal{O}(G)^*$ and $x \in V$ with the relations

$$x \cdot \mu = \sum_i \bigl(v_i^*, gx\bigr)\mu \cdot v_i \quad \text{for all } x \in V,\ \mu \in \mathcal{O}(G)^*,$$

where $\{v_i\}$ is a basis of V and $\{v_i^*\}$ the dual basis of V^*, while $(v_i^*, gx)\mu$ denotes the product of the regular function (v_i^*, gx) and the distribution μ.

We will denote the vector space of length N columns by V_N, so that there are natural actions of $\mathrm{GL}_N, \mathrm{Sp}_N, \mathrm{SO}_N$ on V_N. Let us also denote the action of $g \in G$ on $x \in V$ by x^g.

1.2. Continuous Hecke algebras

We recall the definition of the continuous Hecke algebras of (G, V) following [5].

Given a reductive algebraic group G, its finite dimensional algebraic representation V and a skew-symmetric G-equivariant $\mathbb{C}$-linear map $\kappa : V \times V \to \mathcal{O}(G)^*$, we set

$$\mathcal{H}_\kappa(G, V) := \mathcal{O}(G)^* \ltimes TV/\bigl([x, y] - \kappa(x, y) \mid x, y \in V\bigr).$$

Consider an algebra filtration on $\mathcal{H}_\kappa(G, V)$ by setting $\deg(V) = 1$ and $\deg(\mathcal{O}(G)^*) = 0$.

Definition 1.1. (See [5].) We say that $\mathcal{H}_\kappa(G, V)$ *satisfies the PBW property* if the natural surjective map $\mathcal{O}(G)^* \ltimes SV \twoheadrightarrow \mathrm{gr}\,\mathcal{H}_\kappa(G, V)$ is an isomorphism, where SV denotes the symmetric algebra of V. We call these $\mathcal{H}_\kappa(G, V)$ the *continuous Hecke algebras* of (G, V).

According to [5, Theorem 2.4], $\mathcal{H}_\kappa(G, V)$ satisfies the PBW property if and only if κ satisfies the *Jacobi identity*:

$$\bigl(z - z^g\bigr)\kappa(x, y) + \bigl(y - y^g\bigr)\kappa(z, x) + \bigl(x - x^g\bigr)\kappa(y, z) = 0 \quad \text{for all } x, y, z \in V. \qquad (\dagger)$$

Define the closed subscheme $\Phi \subset G$ by the equation $\wedge^3(1-g_{|_V})=0$. The set of closed points of Φ is the set $S=\{g\in G : \mathrm{rk}(1-g_{|_V})\leq 2\}$. We have:

Proposition 1.1. *(See [5, Proposition 2.8].) If the PBW property holds for $\mathcal{H}_\kappa(G,V)$, then $\kappa(x,y)$ is supported on the scheme Φ for all $x,y\in V$.*

The classification of all κ satisfying (†) was obtained in [5] for the following two cases:

- for the pairs $(G,\mathfrak{h}\oplus\mathfrak{h}^*)$ with $\mathfrak{h}$ being an irreducible faithful G-representation of real or complex type (see [5, Theorem 3.5]),
- for the pair $(\mathrm{Sp}_{2n},V_{2n})$ (see [5, Theorem 3.14]).

For general continuous Hecke algebras such a classification is not known at the moment. However, a particular family of those was established in [5, Theorem 2.13]:

Proposition 1.2. *For any $\tau\in(\mathcal{O}(\mathrm{Ker}\,\rho)^*\otimes\wedge^2V^*)^G$ and $\upsilon\in(\mathcal{O}(\Phi)^*\otimes\wedge^2V^*)^G$, the pairing $\kappa_{\tau,\upsilon}(x,y):=\tau(x,y)+\upsilon((1-g)x,(1-g)y)$ satisfies the Jacobi identity.*

Our first result is a full classification of all κ satisfying (†) for the case of (SO_N,V_N), which is similar to the aforementioned classification for $(\mathrm{Sp}_{2n},V_{2n})$. But it turns out that Φ is not reduced in this case and so we need a more detailed argument.

Theorem 1.3. *The PBW property holds for $\mathcal{H}_\kappa(\mathrm{SO}_N,V_N)$ if and only if there exists an SO_N-invariant distribution $c\in\mathcal{O}(S)^*$ such that $\kappa(x,y)=((g-g^{-1})x,y)c$ for all $x,y\in V_N$.*

The proof of this theorem is presented in Section 2.

1.3. Infinitesimal Hecke algebras

For any triple $(\mathfrak{g},V,\kappa)$ of a Lie algebra $\mathfrak{g}$, its representation V and a $\mathfrak{g}$-equivariant $\mathbb{C}$-bilinear pairing $\kappa:\wedge^2V\to U(\mathfrak{g})$, we define

$$H_\kappa(\mathfrak{g},V):=U(\mathfrak{g})\ltimes TV/\big([x,y]-\kappa(x,y)\mid x,y\in V\big).$$

Endow this algebra with a filtration by setting $\deg(V)=1,\ \deg(\mathfrak{g})=0$.

Definition 1.2. (See [5, Section 4].) We call this algebra the *infinitesimal Hecke algebra* of $(\mathfrak{g},V)$ if it satisfies the *PBW property*, that is, the natural surjective map $U(\mathfrak{g})\ltimes SV\twoheadrightarrow \mathrm{gr}H_\kappa(\mathfrak{g},V)$ is an isomorphism.

Any such algebra gives rise to a continuous Hecke algebra

$$\mathcal{H}_\kappa(G,V):=\mathcal{O}(G)^*\otimes_{U(\mathfrak{g})}H_\kappa(\mathfrak{g},V),$$

where $U(\mathfrak{g})$ is identified with the subalgebra $\mathcal{O}(G)^*_{1_G}\subset\mathcal{O}(G)^*$, consisting of all algebraic distributions set-theoretically supported at $1_G\in G$.

In particular, having a full classification of the continuous Hecke algebras of type (G,V) yields a corresponding classification for the infinitesimal Hecke algebras of $(\mathrm{Lie}(G),V)$. The latter classification was determined explicitly for the cases of $(\mathfrak{g},V)=(\mathfrak{gl}_n,V_n\oplus V_n^*),(\mathfrak{sp}_{2n},V_{2n})$ in [5, Theorem 4.2].

To formulate our classification of infinitesimal Hecke algebras $H_\kappa(\mathfrak{so}_N, V_N)$ we define:

- $\gamma_{2j+1}(x,y) \in S(\mathfrak{so}_N) \simeq \mathbb{C}[\mathfrak{so}_N]$ by

$$\left(x, A\left(1+\tau^2A^2\right)^{-1}y\right)\det\left(1+\tau^2A^2\right)^{-1/2} = \sum_{j\ge 0}\gamma_{2j+1}(x,y)(A)\tau^{2j}, \quad A \in \mathfrak{so}_N,$$

  where we formally set $(1+T)^\alpha := 1 + \sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}T^n$ for $\alpha \in \mathbb{R},\ T \in \tau^2\mathbb{C}[\tau^2]$;
- $r_{2j+1}(x,y) \in U(\mathfrak{so}_N)$ to be the symmetrization of $\gamma_{2j+1}(x,y) \in S(\mathfrak{so}_N)$.

The following theorem is proved in Section 3:

Theorem 1.4. *The PBW property holds for $H_\kappa(\mathfrak{so}_N, V_N)$ if and only if $\kappa = \sum_{j=0}^{k}\zeta_j r_{2j+1}$ for some non-negative integer k and parameters $\zeta_0, \ldots, \zeta_k \in \mathbb{C}$.*

This theorem is very similar to the analogous results for the pairs $(\mathfrak{gl}_n, V_n \oplus V_n^*)$ and $(\mathfrak{sp}_{2n}, V_{2n})$. We denote the corresponding algebra by $H_\zeta(\mathfrak{so}_N, V_N)$ for κ of the above form.

Remark 1.1. (a) For $\zeta_0 \neq 0$, we have $H_{\zeta_0 r_1}(\mathfrak{so}_N, V_N) \simeq U(\mathfrak{so}_{N+1})$. Thus, for an arbitrary ζ we can regard $H_\zeta(\mathfrak{so}_N, V_N)$ as a deformation of $U(\mathfrak{so}_{N+1})$.
(b) Theorem 1.4 does not hold for $N = 2$, since only half of the infinitesimal Hecke algebras are of the form given in the theorem (algebras $H_\kappa(\mathfrak{so}_2, V_2)$ are the same as $H_{\kappa'}(\mathfrak{gl}_1, V_1 \oplus V_1^*)$).

1.4. W-algebras

Here we recall the definitions of finite W-algebras following [7] (see also [9, Section 1.5]).

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ and $e \in \mathfrak{g}$ be a nonzero nilpotent element. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form $(\,,)$. Let χ be the element of $\mathfrak{g}^*$ corresponding to e and $\mathfrak{z}_\chi$ be the stabilizer of χ in $\mathfrak{g}$ (which is the same as the centralizer of e in $\mathfrak{g}$). Fix an $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{g}$. Then $\mathfrak{z}_\chi$ is $\mathrm{ad}(h)$-stable and the eigenvalues of $\mathrm{ad}(h)$ on $\mathfrak{z}_\chi$ are nonnegative integers. Consider the $\mathrm{ad}(h)$-weight grading on $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}}\mathfrak{g}(i)$, that is, $\mathfrak{g}(i) := \{\xi \in \mathfrak{g} \mid [h,\xi] = i\xi\}$. Equip $\mathfrak{g}(-1)$ with the symplectic form $\omega_\chi(\xi,\eta) := \langle\chi, [\xi,\eta]\rangle$. Fix a Lagrangian subspace $l \subset \mathfrak{g}(-1)$ and set $\mathfrak{m} := \bigoplus_{i\le -2}\mathfrak{g}(i) \oplus l \subset \mathfrak{g}$, $\mathfrak{m}' := \{\xi - \langle\chi,\xi\rangle \mid \xi \in \mathfrak{m}\} \subset U(\mathfrak{g})$.

Definition 1.3. (See [10,7].) The W-algebra associated with e (and l) is the algebra $U(\mathfrak{g},e) := (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}')^{\mathrm{adm}}$ with multiplication induced from $U(\mathfrak{g})$.

Let $\{F_\bullet^{st}\}$ denote the PBW filtration on $U(\mathfrak{g})$, while $U(\mathfrak{g})(i) := \{x \in U(\mathfrak{g}) \mid [h,x] = ix\}$. Define $F_kU(\mathfrak{g}) = \sum_{i+2j\le k}(F_j^{st}U(\mathfrak{g}) \cap U(\mathfrak{g})(i))$ and equip $U(\mathfrak{g},e)$ with the induced filtration, denoted $\{F_\bullet\}$ and referred to as the *Kazhdan* filtration.

One of the key results of [7,10] is a description of the associated graded algebra $\mathrm{gr}_{F_\bullet}U(\mathfrak{g},e)$. Recall that the affine subspace $\mathrm{S}_e := \chi + (\mathfrak{g}/[\mathfrak{g},f])^* \subset \mathfrak{g}^*$ is called the *Slodowy slice*. As an affine subspace of $\mathfrak{g}$, the Slodowy slice S_e coincides with $e + \mathfrak{c}$, where $\mathfrak{c} = \mathrm{Ker}_{\mathfrak{g}}\,\mathrm{ad}(f)$. So we can identify $\mathbb{C}[\mathrm{S}_e] \cong \mathbb{C}[\mathfrak{c}]$ with the symmetric algebra $S(\mathfrak{z}_\chi)$. According to [7, Section 3], algebra $\mathbb{C}[\mathrm{S}_e]$ inherits a Poisson structure from $\mathbb{C}[\mathfrak{g}^*]$ and is also graded with $\deg(\mathfrak{z}_\chi \cap \mathfrak{g}(i)) = i+2$.

Theorem 1.5. *(See [7, Theorem 4.1].) The filtered algebra $U(\mathfrak{g},e)$ does not depend on the choice of l (up to a distinguished isomorphism) and $\mathrm{gr}_{F_\bullet}U(\mathfrak{g},e) \cong \mathbb{C}[\mathrm{S}_e]$ as graded Poisson algebras.*

2. Proof of Theorem 1.3

- *Sufficiency*

Given any $c \in (\mathcal{O}(S)^*)^{\mathrm{SO}_N}$, the formula $\kappa(x,y) := ((g-g^{-1})x, y)c$ defines a skew-symmetric SO_N-equivariant pairing $\kappa : V_N \times V_N \to \mathcal{O}(\mathrm{SO}_N)^*$. For $x,y,z \in V_N$ and $g \in \mathrm{SO}_N$ we define

$$h(x,y,z;g) := \big(z - z^g\big)\big(x^g - x^{g^{-1}}, y\big) + \big(y - y^g\big)\big(z^g - z^{g^{-1}}, x\big) + \big(x - x^g\big)\big(y^g - y^{g^{-1}}, z\big).$$

Lemma 2.1. *We have $h(x,y,z;g) = 0$ for all $x,y,z \in V_N$ and $g \in S$.*

Proof. For any $g \in S$ consider the decomposition $V = V^g \oplus (V^g)^\perp$, where $V^g := \mathrm{Ker}(1-g)$ is a codimension ≤ 2 subspace of V. If either of the vectors x,y,z belongs to V^g, then all the three summands are zero and the result follows. Thus, we can assume $x,y,z \in (V^g)^\perp$. Without loss of generality, we can assume that $z = \alpha x + \beta y$ with $\alpha, \beta \in \mathbb{C}$, since $\dim(V^g)^\perp \leq 2$. Then

$$\begin{aligned} h(x,y,z;g) = \alpha\Big(&\big(x - x^g\big)\big(x^g - x^{g^{-1}}, y\big) + \big(x - x^g\big)\big(y^g - y^{g^{-1}}, x\big) + \big(y - y^g\big)\big(x^g - x^{g^{-1}}, x\big)\Big) \\ &+ \beta\Big(\big(y - y^g\big)\big(x^g - x^{g^{-1}}, y\big) + \big(y - y^g\big)\big(y^g - y^{g^{-1}}, x\big) + \big(x - x^g\big)\big(y^g - y^{g^{-1}}, y\big)\Big). \end{aligned}$$

Clearly, $(x^g - x^{g^{-1}}, x) = (x^g, x) - (x, x^g) = 0$ and $(x^g - x^{g^{-1}}, y) = -(y^g - y^{g^{-1}}, x)$, so that the first sum is zero. Likewise, the second sum is zero. The result follows. □

Since c is scheme-theoretically supported on S, we get $h(x,y,z;g)c = 0$ and so (†) holds.

- *Necessity*

Let $I \subset \mathbb{C}[\mathrm{SO}_N]$ be the defining ideal of Φ, that is, I is generated by 3×3 determinants of $1-g$. Consider a closed subscheme $\bar{\Phi} \subset \mathfrak{so}_N$, defined by the ideal $\bar{I} := (\wedge^3 A) \subset \mathbb{C}[\mathfrak{so}_N]$.

Define $E := \mathrm{Rad}(I)/I$ and $\bar{E} := \mathrm{Rad}(\bar{I})/\bar{I}$. Notice that $\bar{E} \simeq E$, since Φ is reduced in the formal neighborhood of any point $g \neq 1$, while the exponential map defines an isomorphism of formal completions $\exp : \bar{\Phi}^{\wedge_0} \xrightarrow{\sim} \Phi^{\wedge_1}$.

On the other hand, we have a short exact sequence of SO_N-modules

$$0 \to E \to \mathcal{O}(\Phi) \to \mathcal{O}(S) \to 0,$$

inducing the following short exact sequence of vector spaces

$$0 \to \big(\wedge^2 V_N^* \otimes \mathcal{O}(S)^*\big)^{\mathrm{SO}_N} \xrightarrow{\phi} \big(\wedge^2 V_N^* \otimes \mathcal{O}(\Phi)^*\big)^{\mathrm{SO}_N} \xrightarrow{\psi} \big(\wedge^2 V_N^* \otimes E^*\big)^{\mathrm{SO}_N} \to 0. \qquad (\natural)$$

It is easy to deduce the necessity for $\kappa \in \mathrm{Im}(\phi)$ by utilizing the arguments from the proof of [5, Theorem 3.14(ii)]. Combining this observation with Proposition 1.1 and an isomorphism $E \simeq \bar{E}$, it suffices to prove the following result:

Lemma 2.2. (a) *The space $(\wedge^2 V_N^* \otimes \bar{E}^*)^{\mathrm{SO}_N}$ is either zero or one-dimensional.*
(b) *If $(\wedge^2 V_N^* \otimes \bar{E}^*)^{\mathrm{SO}_N} \neq 0$, then there exists $\kappa' \in (\wedge^2 V_N^* \otimes \mathcal{O}(\Phi)^*)^{\mathrm{SO}_N}$ not satisfying (†).*[2]

[2] So that any element of $(\wedge^2 V_N^* \otimes \mathcal{O}(\Phi)^*)^{\mathrm{SO}_N}$ satisfying (†) should be in the image of ϕ.

Notice that the adjoint action of SO_N on $\mathfrak{so}_N$ extends to the action of GL_N by $g.A = gAg^t$ for $A \in \mathfrak{so}_N$, $g \in \mathrm{GL}_N$. This endows $\mathbb{C}[\mathfrak{so}_N]$ with a structure of a GL_N-module and both $\bar{I}, \mathrm{Rad}(\bar{I})$ are GL_N-invariant. The following fact was communicated to us by Steven Sam:

Claim 2.3. *As $\mathfrak{gl}_N$-representations $\bar{E} \simeq \wedge^4 V_N$.*

Let us first deduce Lemma 2.2 from the Claim 2.3.

Proof of Lemma 2.2. (a) The following facts are well-known (see [6, Theorems 19.2, 19.14]):

- the $\mathfrak{so}_{2n+1}$-representations $\{\wedge^i V_{2n+1}\}_{i=0}^n$ are irreducible and pairwise non-isomorphic,
- the $\mathfrak{so}_{2n}$-representation $\wedge^n V_{2n}$ decomposes as $\wedge^n V_{2n} \simeq \wedge_+^n V_{2n} \oplus \wedge_-^n V_{2n}$, and $\mathfrak{so}_{2n}$-representations $\{\wedge^0 V_{2n}, \ldots, \wedge^{n-1} V_{2n}, \wedge_+^n V_{2n}, \wedge_-^n V_{2n}\}$ are irreducible and pairwise non-isomorphic.

Combining these facts with Claim 2.3 and an isomorphism $\wedge^k V_N \simeq \wedge^{N-k} V_N^*$, we get

$$\left(\wedge^2 V_{2n+1}^* \otimes \bar{E}^*\right)^{\mathrm{SO}_{2n+1}} = 0, \quad \text{while } \dim\left(\left(\wedge^2 V_{2n}^* \otimes \bar{E}^*\right)^{\mathrm{SO}_{2n}}\right) = \begin{cases} 1, & n = 3, \\ 0, & n \neq 3. \end{cases}$$

(b) For $N = 6$, any nonzero element of $(\wedge^2 V_6^* \otimes \bar{E}^*)^{\mathrm{SO}_6}$ corresponds to the composition

$$\wedge^2 V_6 \xrightarrow[\varphi]{\sim} \wedge^4 V_6^* \simeq \bar{E}^*.$$

Let $M_4 \subset \mathbb{C}[\mathfrak{so}_N]_2$ be the subspace spanned by the Pfaffians of all 4×4 principal minors. This subspace is GL_6-invariant and $M_4 \simeq \wedge^4 V_6$ as $\mathfrak{gl}_6$-representations. Claim 2.3 and simplicity of the spectrum of the $\mathfrak{gl}_6$-module $\mathbb{C}[\mathfrak{so}_6]$ (see Theorem 2.5 below) imply $M_4 \subset \mathrm{Rad}(\bar{I})$ and $M_4 \cap \bar{I} = 0$. It follows that M_4 corresponds to the copy of $\wedge^4 V_6 \simeq \mathrm{Rad}(\bar{I})/\bar{I}$ from Claim 2.3.

Choose an orthonormal basis $\{y_i\}_{i=1}^6$ of V_6, so that any element $A \in \mathfrak{so}_6$ is skew-symmetric with respect to this basis. We denote the corresponding Pfaffian by $\mathrm{Pf}_{\widehat{i,j}}$ (with a correctly chosen sign).[3] We define $\kappa'(y_i \otimes y_j) \in U(\mathfrak{so}_6)$ to be the symmetrization of $\mathrm{Pf}_{\widehat{i,j}}$. Identifying $U(\mathfrak{so}_6)$ with $S(\mathfrak{so}_6)$ as $\mathfrak{so}_6$-modules, we easily see that $\kappa' : \wedge^2 V_6 \to U(\mathfrak{so}_6)$ is $\mathfrak{so}_6$-invariant.

However, κ' does not satisfy the Jacobi identity. Indeed, let us define $\bar{\kappa}' : V_6 \otimes V_6 \to S(\mathfrak{so}_6)$ by $\bar{\kappa}'(y_i \otimes y_j) = \mathrm{Pf}_{\widehat{i,j}}$. Then for any three different indices i, j, k, the corresponding expressions $\{P_{\widehat{i,j}}, x_k\}, \{P_{\widehat{j,k}}, x_i\}, \{P_{\widehat{k,i}}, x_j\}$ coincide up to a sign and are nonzero. So their sum is also non-zero, implying that (†) fails for κ'. □

Proof of Claim 2.3.

○ *Step 1: Description of* $\mathrm{Rad}(\bar{I})$.

Let $\mathrm{Pf}_{ijkl} \in \mathbb{C}[\mathfrak{so}_N]_2$ be the Pfaffians of the principal 4×4 minors corresponding to the rows/columns $\#i, j, k, l$. It is clear that Pf_{ijkl} vanish at rank ≤ 2 matrices and so $\mathrm{Pf}_{ijkl} \in \mathrm{Rad}(\bar{I})$. A beautiful classical result states that those elements generate $\mathrm{Rad}(\bar{I})$, in fact:

Theorem 2.4. *(See [12, Theorem 6.4.1(b)].) The ideal* $\mathrm{Rad}(\bar{I})$ *is generated by* $\{\mathrm{Pf}_{ijkl} \,|\, i < j < k < l\}$.

○ *Step 2: Decomposition of* $\mathbb{C}[\mathfrak{so}_N]$ *as a* $\mathfrak{gl}_N$*-module.*

Let T be the set of all length $\leq N$ Young diagrams $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$. There is a natural bijection between T and the set of all irreducible finite dimensional polynomial $\mathfrak{gl}_N$-representations. For $\lambda \in T$, we

[3] To make a compatible choice of signs, define $\mathrm{Pf}_{\widehat{i,j}}$ as the derivative of the total Pfaffian Pf along $E_{ij} - E_{ji}$.

denote the corresponding irreducible $\mathfrak{gl}_N$-representation by L_λ. Let T^e be the subset of T consisting of all Young diagrams with even columns.

The following result describes the decomposition of $\mathbb{C}[\mathfrak{so}_N]$ into irreducibles:

Theorem 2.5. *(See [1, Theorem 2.5].) As $\mathfrak{gl}_N$-representations $\mathbb{C}[\mathfrak{so}_N] \simeq S(\wedge^2 V_N) \simeq \bigoplus_{\lambda\in T^e} L_\lambda$.*

For any $\lambda \in T^e$, let $\mathcal{I}_\lambda \subset \mathbb{C}[\mathfrak{so}_N]$ be the ideal generated by $L_\lambda \subset \mathbb{C}[\mathfrak{so}_N]$, while $T^e_\lambda \subset T^e$ be the subset of the diagrams containing λ. The arguments of [1] (see also [3, Theorem 5.1]) imply that $\mathcal{I}_\lambda \simeq \bigoplus_{\mu\in T^e_\lambda} L_\mu$ as $\mathfrak{gl}_N$-modules.

∘ *Step 3:* $\mathrm{Rad}(\bar{I})$ *and* $\bar{I}$ *as* $\mathfrak{gl}_N$*-representations.*

Since the subspace $M_4 \subset \mathbb{C}[\mathfrak{so}_N]$, spanned by Pf_{ijkl}, is $\mathfrak{gl}_N$-invariant and is isomorphic to $\wedge^4 V_N$, the results of the previous steps imply that $\mathrm{Rad}(\bar{I}) \simeq \bigoplus_{\mu\in T^e_{(1^4)}} L_\mu$ as $\mathfrak{gl}_N$-modules.

Let $N_3 \subset \mathbb{C}[\mathfrak{so}_N]_3$ be the subspace spanned by the determinants of all 3×3 minors. This is a $\mathfrak{gl}_N$-invariant subspace.

Lemma 2.6. *We have $N_3 \simeq L_{(2^2,1^2)} \oplus L_{(1^6)}$ as $\mathfrak{gl}_N$-representations.*

Proof. According to Step 2, we have $\mathbb{C}[\mathfrak{so}_N]_3 \simeq L_{(1^6)} \oplus L_{(2^2,1^2)} \oplus L_{(3^2)}$. Since the space of 3×3 minors identically vanishes when $N = 2$, and the Schur functor (3,3) does not, it rules $L_{(3^2)}$ out. Also, the space of 3×3 minors is nonzero for $N = 4$, while the Schur functor (1^6) vanishes, so $N_3 \not\simeq L_{(1^6)}$. Since partition (1^6) corresponds to the subspace $M_6 \subset \mathbb{C}[\mathfrak{so}_N]$ spanned by 6×6 Pfaffians, it suffices to prove that $M_6 \subset N_3$. The latter is sufficient to verify for $N = 6$, that is, the Pfaffian Pf of a 6×6 matrix is a linear combination of its 3×3 determinants.[4]

Let $\det^{pqs}_{ijk}$ be the determinant of the 3×3 minor, obtained by intersecting rows $\#i, j, k$ and columns $\#p, q, s$. The following identity is straightforward:

$$-4\,\mathrm{Pf} = -\det^{456}_{123} + \det^{356}_{124} - \det^{346}_{125} + \det^{345}_{126} - \det^{256}_{134} + \det^{246}_{135} - \det^{245}_{136} - \det^{236}_{145} + \det^{235}_{146} - \det^{234}_{156}.$$

This completes the proof of the lemma. □

The results of Step 2 imply that $\bar{I} \simeq \bigoplus_{\mu\in T^e_{(2^2,1^2)}\cup T^e_{(1^6)}} L_\mu$ as $\mathfrak{gl}_N$-modules.

Claim 2.3 follows from the aforementioned descriptions of $\mathfrak{gl}_N$-modules $\bar{I}$ and $\mathrm{Rad}(\bar{I})$. □

3. Proof of Theorem 1.4

Let us introduce some notation:

- $K := \mathrm{SO}_N(\mathbb{R})$ (the maximal compact subgroup of $G = \mathrm{SO}_N(\mathbb{C})$),
- $$s_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & \cdots & 0 \\ \sin\theta & \cos\theta & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \in K, \quad \theta \in [-\pi, \pi],$$

[4] The conceptual proof of this fact is as follows. Note that determinants of 3×3 minors of $A \in \mathfrak{so}_6$ are just the matrix elements of $\wedge^3 A$, and $\wedge^3 A$ acts on $\wedge^3 V_6 = \wedge^3_+ V_6 \oplus \wedge^3_- V_6$. It is easy to see that the trace of $\wedge^3 A$ on $\wedge^3_+ V_6$ is nonzero. This provides a cubic invariant for $\mathfrak{so}_6$, which is unique up to scaling (multiple of Pf).

- $S_\theta := \{gs_\theta g^{-1} | g \in K\} \subset K$,
- $S_\mathbb{R} := S \cap K = \bigcup_{\theta\in[0,\pi]} S_\theta$,[5] so that $S_\mathbb{R}/K$ gets identified with $S^1/\mathbb{Z}_2$.

According to Theorem 1.3, there exists a $\mathbb{Z}_2$-invariant $c \in \mathcal{O}_0(S^1)^*$, which is a linear combination of the delta-function δ_0 (at $0 \in S^1$) and its even derivatives $\delta_0^{(2k)}$, such that[6]

$$\kappa(x, y) = \int_{-\pi}^{\pi} c(\theta)\Big(\int_{S_\theta} \big((g - g^{-1})x, y\big)\, dg\Big) d\theta \quad \text{for all } x, y \in V_N.$$

For $g \in S_\mathbb{R}$ we define a 2-dimensional subspace $V_g \subset V_N$ by $V_g := \mathrm{Im}(1 - g)$. To evaluate the above integral, choose length 1 orthogonal vectors $p, q \in V_g$ such that the restriction of g to V_g is given by the matrix $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ in the basis $\{p, q\}$.

Let us define $J_{p,q} := q \otimes p^t - p \otimes q^t \in \mathfrak{so}_N(\mathbb{R})$. We have:

- $((g - g^{-1})x, y) = 2\sin\theta \cdot (x, J_{p,q}y)$,
- $g = \exp(\theta J_{p,q})$, since $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) = \exp\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

As a result, we get[7]:

$$\kappa(x, y) = \int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} (x, J_{p,q}y)\Big(\int_{-\pi}^{\pi} 2c(\theta)\sin\theta \cdot e^{\theta J_{p,q}}\, d\theta\Big)\, dq\, dp, \tag{1}$$

where S^{N-1} is the unit sphere in $\mathbb{R}^N$ centered at the origin and $S^{N-2}(p)$ is the unit sphere in $\mathbb{R}^{N-1}(p) \subset \mathbb{R}^N$, the hyperplane orthogonal to the line passing through p and the origin.

Since $c(\theta)$ is an arbitrary linear combination of the delta-function and its even derivatives, the above integral is a linear combination of the following integrals:

$$\int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} (x, J_{p,q}y) \cdot J_{p,q}^{2k+1}\, dq\, dp, \quad k \geq 0.$$

This is a standard integral (see [5, Section 4.2] for the analogous calculations). Identifying $U(\mathfrak{so}_N)$ with $S(\mathfrak{so}_N)$ via the symmetrization map, it suffices to compute the integral

$$I_{m;x,y}(A) = \int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} (x, J_{p,q}y) \cdot \mathrm{tr}(AJ_{p,q})^m\, dq\, dp, \quad A \in \mathfrak{so}_N(\mathbb{R}).$$

To compute this expression we introduce

$$F_m(A) := \int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} \mathrm{tr}(AJ_{p,q})^{m+1}\, dq\, dp = \int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} \big(2(Aq, p)\big)^{m+1}\, dq\, dp,$$

so that the former integral can be expressed in the following way:

$$dF_m(A)\big(x \otimes y^t - y \otimes x^t\big) = -2(m + 1)I_{m;x,y}(A).$$

[5] Note that S_θ and $S_{-\theta}$ coincide for $N \geq 3$. That explains why $\theta \in [0, \pi]$ instead of $\theta \in [-\pi, \pi]$.
[6] Here we integrate over the whole circle S^1 instead of $S^1/\mathbb{Z}_2$, but we require $c(\theta) = c(-\theta)$.
[7] Generally speaking, the integration should be taken over the Grassmannian $G_2(\mathbb{R}^N)$. However, it is easier to integrate over the Stiefel manifold $V_2(\mathbb{R}^N)$, which is a principal O(2)-bundle over $G_2(\mathbb{R}^N)$.

Now we compute $F_m(A)$. Notice that

$$G_m(A,\zeta) := \int\limits_{p\in\mathbb{R}^N}\int\limits_{q\in\mathbb{R}^{N-1}(p)} \big(2(Aq,p)\big)^{m+1} e^{-\zeta(p,p)-\zeta(q,q)}\,dq\,dp$$
$$= \int\limits_0^\infty\int\limits_0^\infty e^{-\zeta r_1^2-\zeta r_2^2}\int\limits_{|p|=r_1}\int\limits_{|q|=r_2}\big(2(Aq,p)\big)^{m+1}\,dq\,dp\,dr_2\,dr_1$$
$$= \int\limits_0^\infty\int\limits_0^\infty e^{-\zeta r_1^2-\zeta r_2^2} r_1^{m+N} r_2^{m+N-1}\,dr_2\,dr_1\cdot F_m(A) = K_{m+N}(\zeta)K_{m+N-1}(\zeta)F_m(A),$$

where

$$K_l(\zeta) := \int\limits_0^\infty e^{-\zeta r^2} r^l\,dr = \begin{cases} \frac{k!}{2\zeta^{k+1}}, & l=2k+1,\\ \frac{(2k-1)!!\sqrt{\pi}}{2^{k+1}\zeta^{k+1/2}}, & l=2k.\end{cases}$$

As a result, we get

$$G_m(A,\zeta) = \frac{\sqrt{\pi}(m+N-1)!}{2^{m+N+1}\zeta^{m+N+1/2}}F_m(A).$$

On the other hand, we have:

$$\sum_{m=-1}^{\infty}\frac{1}{(m+1)!}G_m(A,\zeta) = \int\limits_{p\in\mathbb{R}^N}\int\limits_{q\in\mathbb{R}^{N-1}(p)} e^{2(Aq,p)}e^{-\zeta(p,p)-\zeta(q,q)}\,dq\,dp$$
$$= \int\limits_{p\in\mathbb{R}^N} e^{-\zeta(p,p)}\int\limits_{q\in\mathbb{R}^{N-1}(p)} e^{-2(q,Ap)-\zeta(q,q)}\,dq\,dp \overset{q':=q+\frac{Ap}{\zeta}}{=} \int\limits_{p\in\mathbb{R}^N} e^{-\zeta(p,p)}\int\limits_{q'\in\mathbb{R}^{N-1}(p)} e^{-\zeta(q',q')}e^{\frac{1}{\zeta}(Ap,Ap)}\,dq'\,dp$$
$$= \int\limits_{p\in\mathbb{R}^N} e^{-\zeta(p,p)+\frac{1}{\zeta}(Ap,Ap)}\,dp\cdot(\pi/\zeta)^{\frac{N-1}{2}} = (\pi/\zeta)^{\frac{N-1}{2}}\int\limits_{p\in\mathbb{R}^N} e^{((-\zeta-\frac{1}{\zeta}A^2)p,p)}\,dp$$
$$= \frac{\pi^{N-\frac12}}{\zeta^{\frac{N-1}{2}}}\det\Big(\zeta+\frac{1}{\zeta}A^2\Big)^{-1/2} = \frac{\pi^{N-\frac12}}{\zeta^{N-\frac12}}\det\big(1+\zeta^{-2}A^2\big)^{-1/2}.$$

Hence, $F_m(A)$ is equal to a constant times the coefficient of τ^{m+1} in $\det(1+\tau^2A^2)^{-1/2}$, expanded as a power series in τ. Differentiating $\det(1+\tau^2A^2)^{-1/2}$ along $B\in\mathfrak{so}_N$, we get

$$\frac{\partial}{\partial B}\big(\det\big(1+\tau^2A^2\big)^{-1/2}\big) = -\frac{\tau^2\operatorname{tr}(BA(1+\tau^2A^2)^{-1})}{\det(1+\tau^2A^2)^{1/2}}.$$

Setting $B = x\otimes y^t - y\otimes x^t$ yields $2\tau^2(x,A(1+\tau^2A^2)^{-1}y)\det(1+\tau^2A^2)^{-1/2}$ as desired. □

4. Poisson center of algebras $H_\zeta^{\mathrm{cl}}(\mathfrak{so}_N)$

Following [4], we introduce the Poisson algebras $H_\zeta^{\mathrm{cl}}(\mathfrak{so}_N, V_N)$, where $\zeta = (\zeta_0,\dots,\zeta_k)$ is a deformation parameter. As algebras these are $S(\mathfrak{so}_N\oplus V_N)$ with a Poisson bracket $\{\cdot,\cdot\}$ modeled after the commutator

$[\cdot,\cdot]$ of $H_\zeta(\mathfrak{so}_N, V_N)$, that is, $\{x, y\} = \sum_j \zeta_j \gamma_{2j+1}(x, y)$. We prefer the following short formula for $\{\cdot,\cdot\} : V_N \times V_N \to \mathbb{C}[\mathfrak{so}_N] \simeq S(\mathfrak{so}_N)$:

$$\{x, y\} = \mathrm{Res}_{z=0}\, \zeta\big(z^{-2}\big)\big(x, A\big(1 + z^2A^2\big)^{-1}y\big) \det\big(1 + z^2A^2\big)^{-1/2} z^{-1}dz, \quad \forall x, y \in V_N, A \in \mathfrak{so}_N, \qquad (*)$$

where $\zeta(z) := \sum_{i\geq 0} \zeta_i z^i$ is the generating function of the deformation parameters.

In fact, we can view algebras $H_\zeta(\mathfrak{so}_N, V_N)$ as *quantizations* of the algebras $H^{\mathrm{cl}}_\zeta(\mathfrak{so}_N, V_N)$. The latter algebras still carry some important information. The main result of this section is a computation of their Poisson center $\mathfrak{z}_{\mathrm{Pois}}(H^{\mathrm{cl}}_\zeta(\mathfrak{so}_N, V_N))$.

Let us first recall the corresponding result in the non-deformed case ($\zeta = 0$), when the corresponding algebra is just $S(\mathfrak{so}_N \ltimes V_N)$ with a Lie–Poisson bracket. To state the result we introduce some more notation:

- Define $p_i(A) \in \mathbb{C}$ via $\det(I_N + tA) = \sum_{j=0}^N p_j(A)t^j$ for $A \in \mathfrak{gl}_N$.
- Define $b_i(A) \in \mathfrak{gl}_N$ via $b_0(A) = I_N,\ b_k(A) = \sum_{j=0}^k (-1)^j p_j(A)A^{k-j}$ for $k > 0$.
- Define $\mathfrak{a}_N := \mathfrak{so}_N \ltimes V_N$; we identify $\mathfrak{a}^*_N$ with $\mathfrak{a}_N$ via the natural pairing.
- Define $\psi_k : \mathfrak{a}^*_N \to \mathbb{C}$ by $\psi_k(A, v) = (v, b_{2k}(A)v)$ for $A \in \mathfrak{so}_N$, $v \in V_N$, $k \geq 0$.
- If $N = 2n + 1$, ψ_n is actually the square of a polynomial function $\widehat{\psi}_n$, which can be realized explicitly as the Pfaffian of the matrix $\left(\begin{smallmatrix} A & v \\ -v^t & 0 \end{smallmatrix}\right) \in \mathfrak{so}_{2n+2}$.
- Identifying $\mathbb{C}[\mathfrak{a}^*_N] \simeq S(\mathfrak{a}_N)$, let $\tau_k \in S(\mathfrak{a}_N)$ (respectively $\widehat{\tau}_{n+1} \in S(\mathfrak{a}_{2n+1})$) be the elements corresponding to ψ_{k-1} (respectively $\widehat{\psi}_n$).

The following result is due to [11, Sections 3.7, 3.8]:

Proposition 4.1. *Let $\mathfrak{z}_{\mathrm{Pois}}(A)$ denote the Poisson center of the Poisson algebra A. We have:*
(a) *$\mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{a}_{2n}))$ is a polynomial algebra in free generators $\{\tau_1, \ldots, \tau_n\}$.*
(b) *$\mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{a}_{2n+1}))$ is a polynomial algebra in free generators $\{\tau_1, \ldots, \tau_n, \widehat{\tau}_{n+1}\}$.*

Similarly to the cases of $\mathfrak{gl}_n, \mathfrak{sp}_{2n}$, this result can be generalized for arbitrary deformations ζ. In fact, for any deformation parameter $\zeta = (\zeta_0, \ldots, \zeta_k)$ the Poisson center $\mathfrak{z}_{\mathrm{Pois}}(H^{\mathrm{cl}}_\zeta(\mathfrak{so}_N, V_N))$ is still a polynomial algebra in $\lfloor \frac{N+1}{2} \rfloor$ generators. This is established in the following theorem:

Theorem 4.2. *Define $c_i \in \mathbb{C}[\mathfrak{so}_N]^{\mathrm{SO}_N} \simeq \mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{so}_N))$ via $\sum_i (-1)^i c_i t^{2i} = c(t)$, where*

$$c(t) := \mathrm{Res}_{z=0}\, \zeta\big(z^{-2}\big) \frac{\det(1 + t^2A^2)^{1/2}}{\det(1 + z^2A^2)^{1/2}} \frac{z^{-1}dz}{1 - t^{-2}z^2}.$$

(a) *$\mathfrak{z}_{\mathrm{Pois}}(H^{\mathrm{cl}}_\zeta(\mathfrak{so}_{2n}, V_{2n}))$ is a polynomial algebra in free generators $\{\tau_1 + c_1, \ldots, \tau_n + c_n\}$.*
(b) *$\mathfrak{z}_{\mathrm{Pois}}(H^{\mathrm{cl}}_\zeta(\mathfrak{so}_{2n+1}, V_{2n+1}))$ is a polynomial algebra in free generators $\{\tau_1 + c_1, \ldots, \tau_n + c_n, \widehat{\tau}_{n+1}\}$.*

Let us introduce some more notation before proceeding to the proof:

- Let $\{x_i\}_{i=1}^N$ be a basis of V_N such that $(x_i, x_j) = \delta^j_{N+1-i}$.
- Let $J = (J_{ij})_{i,j=1}^N$ be the corresponding *anti-diagonal* symmetric matrix, i.e., $J_{ij} = \delta^j_{N+1-i}$.
  Notice that $A = (a_{ij}) \in \mathfrak{so}_N$ if and only if $a_{ij} = -a_{N+1-j,N+1-i}$ for all i, j.
- Let $\mathfrak{h}_N$ be the Cartan subalgebra of $\mathfrak{so}_N$ consisting of the diagonal matrices.
- Define $e_{(i,j)} := E_{i,j} - E_{N+1-j,N+1-i} \in \mathfrak{so}_N$ for $i, j \leq N$ (in particular, $e_{(i,N+1-i)} = 0\ \forall i$).
- We set $e_i := e_{(i,i)}$ for $1 \leq i \leq n := \lfloor \frac{N}{2} \rfloor$, so that $\{e_i\}_{i=1}^n$ form a basis of $\mathfrak{h}_N$.
- Define symmetric polynomials $\sigma_i \in \mathbb{C}[z_1, \ldots, z_n]^{S_n}$ via $\prod_{i=1}^n (1 + tz_i) = \sum_{i=0}^n t^i \sigma_i(z_1, \ldots, z_n)$.

Proof of Theorem 4.2. We shall show that the elements $\tau_i + c_i$ (and $\widehat{\tau}_{n+1}$ for $N = 2n + 1$) are Poisson central. Combined with Proposition 4.1 this clearly implies the result by a deformation argument. Since

$\{\tau_i, \mathfrak{so}_N\} = 0$ for $\zeta = 0$, we still have $\{\tau_i, \mathfrak{so}_N\} = 0$ for arbitrary ζ. This implies $\{\tau_i + c_i, \mathfrak{so}_N\} = 0$ as $c_i \in \mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{so}_N))$. Therefore we just need to verify

$$\{c_i, x_q\} = -\{\tau_i, x_q\} \quad \text{for all } 1 \le q \le N. \tag{2}$$

Using $\psi_s(A, v) = (v, b_{2s}(A)v) = \sum_{k,l=1}^{N} x_k x_l b_{2s}(A)_{N+1-k,l}$, we get:

$$\{\tau_{s+1}, x_q\} = \sum_{k,l} \{b_{2s}(A)_{N+1-k,l}, x_q\} x_k x_l + \sum_{k,l} b_{2s}(A)_{N+1-k,l} \{x_k, x_q\} x_l + \sum_{k,l} b_{2s}(A)_{N+1-k,l} x_k \{x_l, x_q\}.$$

The first summand is zero due to Proposition 4.1. On the other hand, $AJ + JA^t = 0$ implies $(A^{2j})_{N+1-k,l} = (A^{2j})_{N+1-l,k}$ and $p_{2j+1}(A) = 0$ for all $j \ge 0$. Hence,

$$b_{2s}(A) = A^{2s} + p_2(A)A^{2s-2} + p_4(A)A^{2s-4} + \ldots + p_{2s}(A), \qquad b_{2s}(A)_{n+1-k,l} = b_{2s}(A)_{n+1-l,k}.$$

Combining this with $\{c_{s+1}, x_q\} = \sum_{p \ne N+1-q} \frac{\partial c_{s+1}}{\partial e_{(p,q)}} x_p$, we see that (2) is equivalent to:

$$\frac{\partial c_{s+1}}{\partial e_{(p,q)}} = -2 \sum_{l} b_{2s}(A)_{N+1-p,l} \operatorname{Res}_{z=0} \zeta\big(z^{-2}\big) \frac{(x_l, A(1+z^2A^2)^{-1}x_q)}{\det(1+z^2A^2)^{1/2}} \frac{dz}{z} \quad \text{for all } p, q \le N. \tag{3}$$

Because both sides of (3) are SO_N-invariant, it suffices to verify (3) for $A \in \mathfrak{h}_N$, that is, for

- $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)$ in the case $N = 2n$,
- $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n, 0, -\lambda_n, \ldots, -\lambda_1)$ in the case $N = 2n+1$.

For $p \ne q$, both sides of (3) are zero. For $p = q \le n$, the only nonzero summand on the right hand side of (3) is the one corresponding to $l = N+1-q$. In this case:

$$b_{2s}(A)_{N+1-q,N+1-q} = \lambda_q^{2s} - \sigma_1\big(\lambda_1^2, \ldots, \lambda_n^2\big)\lambda_q^{2s-2} + \ldots + (-1)^s \sigma_s\big(\lambda_1^2, \ldots, \lambda_n^2\big) = (-1)^s \frac{\partial \sigma_{s+1}(\lambda_1^2, \ldots, \lambda_n^2)}{\partial \lambda_q^2},$$

while $(x_{N+1-q}, A(1+z^2A^2)^{-1}x_q) = \frac{\lambda_q}{1+z^2\lambda_q^2}$ and $\det(1+z^2A^2)^{1/2} = \prod_{i=1}^{n}(1+z^2\lambda_i^2)$.

For $p = q > \lfloor \frac{N+1}{2} \rfloor$, we get the same equalities with $\lambda_i \leftrightarrow -\lambda_i$. As a result, (3) is equivalent to:

$$\frac{\partial c_{s+1}(\lambda_1, \ldots, \lambda_n)}{\partial \lambda_q^2} = (-1)^{s+1} \frac{\partial \sigma_{s+1}(\lambda_1^2, \ldots, \lambda_n^2)}{\partial \lambda_q^2} \operatorname{Res}_{z=0} \zeta\big(z^{-2}\big) \frac{z^{-1}dz}{(1+z^2\lambda_q^2)\prod_{i=1}^{n}(1+z^2\lambda_i^2)}.$$

We thus need to verify the following identities for $c(t)$:

$$\frac{\partial c(t)}{\partial \lambda_q^2} = \frac{\partial \prod_{i=1}^{n}(1+t^2\lambda_i^2)}{\partial \lambda_q^2} \operatorname{Res}_{z=0} \frac{\zeta(z^{-2})z^{-1}dz}{(1+z^2\lambda_q^2)\prod_{i=1}^{n}(1+z^2\lambda_i^2)}. \tag{4}$$

This is a straightforward verification and we leave it to an interested reader. This proves that $\tau_i + c_i \in \mathfrak{z}_{\mathrm{Pois}}(H_\zeta^{\mathrm{cl}}(\mathfrak{so}_N, V_N))$ for all $1 \le i \le n$. For $N = 2n+1$, we also get a Poisson-central element $\tau_{n+1} + c_{n+1}$. Since $c_{n+1} = 0$, we have

$$\widehat{\tau}_{n+1}^2 = \tau_{n+1} \in \mathfrak{z}_{\mathrm{Pois}}\big(H_\zeta^{\mathrm{cl}}(\mathfrak{so}_{2n+1}, V_{2n+1})\big) \Rightarrow \widehat{\tau}_{n+1} \in \mathfrak{z}_{\mathrm{Pois}}\big(H_\zeta^{\mathrm{cl}}(\mathfrak{so}_{2n+1}, V_{2n+1})\big).$$

This completes the proof of the theorem. □

Definition 4.1. The element $\tau_1' = \tau_1 + c_1$ is called the *Poisson Casimir element* of $H_\zeta^{\mathrm{cl}}(\mathfrak{so}_N, V_N)$.

As a straightforward consequence of Theorem 4.2, we get:

Corollary 4.3. *We have* $\tau_1' = \tau_1 + \sum_{j=0}^{k}(-1)^{j+1}\zeta_j \operatorname{tr} S^{2j+2}A$.

5. The key isomorphism

5.1. Algebras $H_m(\mathfrak{so}_N, V_N)$

Let us first introduce the universal infinitesimal Hecke algebras of $(\mathfrak{so}_N, V_N)$:

Definition 5.1. Define the *universal length m infinitesimal Hecke algebra* $H_m(\mathfrak{so}_N, V_N)$ as

$$H_m(\mathfrak{so}_N, V_N) := U(\mathfrak{so}_N) \ltimes T(V_N)[\zeta_0, \dots, \zeta_{m-1}] \Big/ \left([A, x] - A(x), [x, y] - \sum_{j=0}^{m-1} \zeta_j r_{2j+1}(x, y) - r_{2m+1}(x, y)\right),$$

where $A \in \mathfrak{so}_N$, $x, y \in V_N$ and $\{\zeta_i\}_{i=0}^{m-1}$ are central. The filtration is induced from the grading on $T(\mathfrak{so}_N \oplus V_N)[\zeta_0, \dots, \zeta_{m-1}]$ with $\deg(\mathfrak{so}_N) = 2$, $\deg(V_N) = 2m + 2$ and $\deg(\zeta_i) = 4(m - i)$.

The algebra $H_m(\mathfrak{so}_N, V_N)$ is free over $\mathbb{C}[\zeta_0, \dots, \zeta_{m-1}]$ and $H_m(\mathfrak{so}_N, V_N)/(\zeta_i - c_i)_{i=0}^{m-1}$ is the usual infinitesimal Hecke algebra $H_{\zeta_c}(\mathfrak{so}_N, V_N)$ for $\zeta_c = c_0 r_1 + \ldots + c_{m-1} r_{2m-1} + r_{2m+1}$.

Remark 5.1. For an $\mathfrak{so}_N$-equivariant pairing $\eta : \wedge^2 V_N \to U(\mathfrak{so}_N)[\zeta_0, \dots, \zeta_{m-1}]$ such that $\deg(\eta(x, y)) \le 4m+2$, the algebra $U(\mathfrak{so}_N) \ltimes T(V_N)[\zeta_0, \dots, \zeta_{m-1}]/([A, x]-A(x), [x, y]-\eta(x, y))$ satisfies the PBW property if and only if $\eta(x, y) = \sum_{i=0}^{m} \eta_i r_{2i+1}(x, y)$ with $\eta_i \in \mathbb{C}[\zeta_0, \dots, \zeta_{m-1}]$ degree $\le 4(m - i)$ polynomials (this is completely analogous to Theorem 1.4).

5.2. Isomorphisms $\bar{\Theta}$ and $\bar{\Theta}^{\mathrm{cl}}$

The main goal of this section it to establish an abstract isomorphism between the algebras $H_m(\mathfrak{so}_N, V_N)$ and the W-algebras $U(\mathfrak{so}_{N+2m+1}, e_m)$, where $e_m \in \mathfrak{so}_{N+2m+1}$ is a nilpotent element of the Jordan type $(1^N, 2m + 1)$. We make a particular choice of such an element[8]:

- $e_m := \sum_{j=1}^{m} E_{N+j,N+j+1} - \sum_{j=1}^{m} E_{N+m+j,N+m+j+1}$.

Recall the Lie algebra inclusion $\iota : \mathfrak{q} \hookrightarrow U(\mathfrak{g}, e)$ from [9, Section 1.6], where $\mathfrak{q} := \mathfrak{z}_\mathfrak{g}(e, h, f)$. For $(\mathfrak{g}, e) = (\mathfrak{so}_{N+2m+1}, e_m)$ we have $\mathfrak{q} \simeq \mathfrak{so}_N$. We will also denote the corresponding centralizer of $e_m \in \mathfrak{so}_{N+2m+1}$ and the Slodowy slice by $\mathfrak{z}_{N,m}$ and $\mathrm{S}_{N,m}$, respectively.

Theorem 5.1. *For $m \ge 1$, there is a unique isomorphism $\bar{\Theta} : H_m(\mathfrak{so}_N, V_N) \xrightarrow{\sim} U(\mathfrak{so}_{N+2m+1}, e_m)$ of filtered algebras such that $\bar{\Theta}\,|_{\mathfrak{so}_N} = \iota\,|_{\mathfrak{so}_N}$.*

[8] In this section, we view $\mathfrak{so}_N$ as corresponding to the pair $(V_N, (\cdot,\cdot))$, where $(\cdot,\cdot)$ is represented by the symmetric matrix $J' = (J'_{ij})$ with $J'_{ij} = \delta_i^j$, $J'_{i,N+k} = J'_{N+k,i} = 0$, $J'_{N+k,N+l} = \delta_{k+l}^{2m+2}$, $\forall i, j \le N$, $k, l \le 2m + 1$.

Sketch of the proof. Notice that $\mathfrak{z}_{N,m} \simeq \mathfrak{so}_N \oplus V_N \oplus \mathbb{C}^m$ as vector spaces, where $\mathfrak{so}_N \simeq \mathfrak{q} = \mathfrak{z}_{N,m}(0), V_N \subset \mathfrak{z}_{N,m}(2m)$ and $\mathbb{C}^m$ has a basis $\{\xi_0, \ldots, \xi_{m-1}\}$ with $\xi_i \in \mathfrak{z}_{N,m}(4m-4i-2)$. Here $\xi_{m-j} = e_m^{2j-1} \in \mathfrak{so}_N$ for $1 \le j \le m$, V_N is embedded via $x_i \mapsto E_{i,N+2m+1} - E_{N+1,i}$, while $\mathfrak{so}_N$ is embedded as a top-left $N \times N$ block of $\mathfrak{so}_{N+2m+1}$.

Let us recall that one of the key ingredients in the proof of [9, Theorem 7] was an additional $\mathbb{Z}$-grading Gr on the corresponding W-algebras.[9] In both cases of $(\mathfrak{sl}_{n+m}, e_m), (\mathfrak{sp}_{2n+2m}, e_m)$ such a grading was induced from the weight-decomposition with respect to $\mathrm{ad}(\iota(h))$, $h \in \mathfrak{q}$.

If $N = 2n$, same argument works for $\mathfrak{g} = \mathfrak{so}_{N+2m+1}$ as well. Namely, consider $h \in \mathfrak{q} \simeq \mathfrak{so}_{2n}$ to be the diagonal matrix $I'_n := \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$. The operator $\mathrm{ad}(\iota(I'_n))$ acts on $\mathfrak{z}_{N,m}$ with zero eigenvalues on $\mathbb{C}^m$, with even eigenvalues on $\mathfrak{so}_N$, and with eigenvalues $\{\pm 1\}$ on V_N.

However, there is no appropriate $h \in \mathfrak{q}$ in the case of $N = 2n+1$. Instead, such a grading originates from the adjoint action of the element

$$\mathrm{g}_0 := (\underbrace{-1, \ldots, -1}_{N}, \underbrace{1, \ldots, 1}_{2m+1}) \in \mathrm{O}(N+2m+1).$$

This element defines a $\mathbb{Z}_2$-grading on $U(\mathfrak{so}_{N+2m+1})$ and further a $\mathbb{Z}_2$-grading Gr on the W-algebra $U(\mathfrak{so}_{N+2m+1}, e_m)$. The induced $\mathbb{Z}_2$-grading Gr' on $\mathrm{gr}U(\mathfrak{so}_{N+2m+1}, e_m) \simeq S(\mathfrak{z}_{N,m})$ satisfies the desired properties, that is, $\deg(\mathbb{C}^m) = 0,\ \deg(\mathfrak{so}_N) = 0,\ \deg(V_N) = 1$.

Therefore the algebra $U(\mathfrak{so}_{N+2m+1}, e_m)$ is equipped both with a Kazhdan filtration and a $\mathbb{Z}_2$-grading Gr. Moreover, the corresponding isomorphism at the Poisson level is established in Theorem 5.2. Now the proof proceeds along the same lines as in [9, Theorem 7]. □

Let us introduce some more notation:

- Let $\bar{\iota} : \mathfrak{so}_N \oplus V_N \oplus \mathbb{C}^m \xrightarrow{\sim} \mathfrak{z}_{N,m}$ denote the isomorphism from the proof of Theorem 5.1.
- Let $H^{\mathrm{cl}}_m(\mathfrak{so}_N, V_N)$ be the Poisson counterpart of $H_m(\mathfrak{so}_N, V_N)$ (compare to algebras $H^{\mathrm{cl}}_\zeta(\mathfrak{so}_N, V_N)$).
- Define $P_j \in \mathbb{C}[\mathfrak{so}_{N+2m+1}]$ by $\det(I_{N+2m+1} + tA) = \sum_{j=0}^{N+2m+1} P_j(A)t^j$.
- Define $\{\bar{\Theta}_i\}_{i=0}^{m-1} \in S(\mathfrak{z}_{N,m}) \simeq \mathbb{C}[\mathrm{S}_{N,m}]$ by $\bar{\Theta}_i := P_{2(m-i)|_{\mathrm{S}_{N,m}}}$.

The following result can be considered as a Poisson version of Theorem 5.1:

Theorem 5.2. *The formulas*

$$\bar{\Theta}^{\mathrm{cl}}(A) = \bar{\iota}(A), \qquad \bar{\Theta}^{\mathrm{cl}}(y) = \frac{(-1)^{\frac{m}{2}}}{2} \cdot \bar{\iota}(y), \qquad \bar{\Theta}^{\mathrm{cl}}(\zeta_k) = (-1)^{m-j}\bar{\Theta}_k$$

define an isomorphism $\bar{\Theta}^{\mathrm{cl}} : H^{\mathrm{cl}}_m(\mathfrak{so}_N, V_N) \xrightarrow{\sim} S(\mathfrak{z}_{N,m}) \simeq \mathbb{C}[\mathrm{S}_{N,m}]$ *of Poisson algebras.*

The proof of this theorem proceeds along the same lines as for $\mathfrak{sp}_{2N}$ (see [9, Theorem 10]).

5.3. Consequences

Let us now deduce a few results on the infinitesimal Hecke algebras of $(\mathfrak{so}_N, V_N)$.

Corollary 5.3. *Poisson varieties corresponding to arbitrary full central reductions of Poisson infinitesimal Hecke algebras* $H^{\mathrm{cl}}_\zeta(\mathfrak{so}_N, V_N)$ *have finitely many symplectic leaves.*

[9] Actually, as exhibited by the case of $\mathfrak{sp}_{2n+2m}$, it suffices to have a $\mathbb{Z}_2$-grading.

Corollary 5.4. (a) *The center* $Z(H_\zeta(\mathfrak{so}_N, V_N))$ *is a polynomial algebra in* $\lfloor \frac{N+1}{2} \rfloor$ *generators.*[10]
(b) *The infinitesimal Hecke algebra* $H_\zeta(\mathfrak{so}_N, V_N)$ *is free over its center* $Z(H_\zeta(\mathfrak{so}_N, V_N))$.
(c) *Full central reductions of* $\mathrm{gr} H_\zeta(\mathfrak{so}_N, V_N)$ *are normal, complete intersection integral domains.*

Finally, the isomorphism of Theorem 5.1 provides the appropriate categories $\mathcal{O}$ for the algebras $H_m(\mathfrak{so}_N, V_N)$ (and hence for $H_\zeta(\mathfrak{so}_N, V_N)$) once we have them for the finite W-algebras. The categories $\mathcal{O}$ for the finite W-algebras were first introduced in [2] and were further studied in [8]. Namely, recall that we have an embedding $\mathfrak{q} \subset U(\mathfrak{g}, e)$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{q}$ and set $\mathfrak{g}_0 := \mathfrak{z}_\mathfrak{g}(\mathfrak{t})$. Pick an integral element $\theta \in \mathfrak{t}$ such that $\mathfrak{z}_\mathfrak{g}(\theta) = \mathfrak{g}_0$. By definition, the category $\mathcal{O}$ (for θ) consists of all finitely generated $U(\mathfrak{g}, e)$-modules M, where the action of $\mathfrak{t}$ is diagonalizable with finite dimensional eigenspaces and, moreover, the set of weights is bounded from above in the sense that there are complex numbers $\alpha_1, \dots, \alpha_k$ such that for any weight λ of M there is i with $\alpha_i - \langle \theta, \lambda \rangle \in \mathbb{Z}_{\leqslant 0}$. The category $\mathcal{O}$ has analogues of Verma modules, $\Delta(N^0)$. Here N^0 is an irreducible module over the W-algebra $U(\mathfrak{g}_0, e)$, where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{t}$. In the case of interest $(\mathfrak{g}, e) = (\mathfrak{so}_{N+2m+1}, e_m)$, we have $\mathfrak{g}_0 = \mathfrak{so}_{2m+1} \times \mathbb{C}^N$ and e is principal in $\mathfrak{g}_0$. In this case, the W-algebra $U(\mathfrak{g}_0, e)$ coincides with the center of $U(\mathfrak{g}_0)$. Therefore N^0 is a one-dimensional space, and the set of all possible N^0 is identified, via the Harish-Chandra isomorphism, with the quotient $\mathfrak{h}^*/W_0$, where $\mathfrak{h}, W_0$ are a Cartan subalgebra and the Weyl group of $\mathfrak{g}_0$ (we take the quotient with respect to the dot-action of W_0 on $\mathfrak{h}^*$). As in the usual BGG category $\mathcal{O}$, each Verma module has a unique irreducible quotient, $L(N^0)$. Moreover, the map $N^0 \mapsto L(N^0)$ is a bijection between the set of finite dimensional irreducible $U(\mathfrak{g}_0, e)$-modules, $\mathfrak{h}^*/W_0$, in our case, and the set of irreducible objects in $\mathcal{O}$. We remark that all finite dimensional irreducible modules lie in $\mathcal{O}$.

6. Casimir element

In this section we determine the first nontrivial central element of the algebras $H_\zeta(\mathfrak{so}_N, V_N)$. In the non-deformed case $\zeta = 0$, we have $t_1 := (v, v) \in Z(H_0(\mathfrak{so}_N, V_N))$. Similarly to Corollary 4.3, this element can be deformed to a central element of $H_\zeta(\mathfrak{so}_N, V_N)$ by adding an element of $Z(U(\mathfrak{so}_N))$.

In order to formulate the result, we introduce some more notation:

- Define $\omega_s := \frac{\pi^{1/2}(s+N-1)!}{2^{s+N+1}}$ and $\mu_s := \pi^{N-\frac{1}{2}}(s+1)!\omega_s^{-1},\ \nu_s := -\frac{\mu_s}{s+1}$.
- For a sequence $\{\zeta_j\}_{j=0}^m$ define $\{a_j\}_{j=0}^m$ recursively via $\zeta_j = 2\nu_{2j+1} \sum_{l=1}^{m+1-j} (-1)^{l+1} \binom{2j+2l}{2l-1} a_{j+l-1}$.
- Define a sequence of parameters $\{g_j\}_{j=1}^{m+1}$ via $g_j = 2\mu_{2j-1}(-2a_{j-1} + \sum_{l=1}^{m+1-j} (-1)^{l+1} \binom{2j+2l}{2l} a_{j+l-1})$.
- Define a polynomial $g(z) := \sum_{j=1}^{m+1} g_j z^j$.
- Define $A(z)(x, y) := (x, A(1 + z^2A^2)^{-1}y) \det(1 + z^2A^2)^{-1/2}$ and $B(z) := \det(1 + z^2A^2)^{-1/2}$.
- Let $[z^m]f(z)$ denote the coefficient of z^m in the series $f(z)$.
- Define $C \in Z(U(\mathfrak{so}_N))$ to be the symmetrization of $\mathrm{Res}_{z=0}\, g(z^{-2}) \det(1 + z^2A^2)^{-1/2} z^{-1} dz$.

Then we have:

Theorem 6.1. *The element* $t_1' := t_1 + C$ *is a central element of* $H_\zeta(\mathfrak{so}_N, V_N)$.

Definition 6.1. We call $t_1' = t_1 + C$ the *Casimir element* of $H_\zeta(\mathfrak{so}_N, V_N)$.

Remark 6.1. The same formula provides a central element of the algebra $H_m(\mathfrak{so}_N, V_N)$, where $C \in Z(U(\mathfrak{so}_N))[\zeta_0, \dots, \zeta_{m-1}]$.

---

[10] Here we use the description of the center of the W-algebras, see [9, Theorem 5] for a reference.

Theorem 6.1 can be used to establish explicitly the isomorphism $\bar{\Theta}$ of Theorem 5.1 in the same way as this has been achieved in [9, Section 4.6] for the $\mathfrak{gl}_n$ case.

Proof of Theorem 6.1. Commutativity of t_1' with $\mathfrak{so}_N$ follows from the following argument:

$$[t_1, \mathfrak{so}_N] = 0 \in H_0(\mathfrak{so}_N, V_N) \quad \Rightarrow \quad [t_1, \mathfrak{so}_N] = 0 \in H_\zeta(\mathfrak{so}_N, V_N) \quad \Rightarrow \quad [t_1', \mathfrak{so}_N] = 0 \in H_\zeta(\mathfrak{so}_N, V_N).$$

Let us now verify $[t_1 + C, x] = 0$ for any $x \in V_N$.

Identifying $U(\mathfrak{so}_N)$ with $S(\mathfrak{so}_N)$ via the symmetrization map and recalling (1), we get:

$$\Big[\sum x_i^2, x\Big] = \sum_i x_i \int\limits_{p\in S^{N-1}} \int\limits_{q\in S^{N-2}(p)} (x_i, J_{p,q}x)\left(\int\limits_{-\pi}^{\pi} 2c(\theta)\sin\theta e^{\theta J_{p,q}} d\theta\right) dqdp$$
$$+ \sum_i \int\limits_{p\in S^{N-1}} \int\limits_{q\in S^{N-2}(p)} \left(\int\limits_{-\pi}^{\pi} 2c(\theta)\sin\theta e^{\theta J_{p,q}} d\theta\right)(x_i, J_{p,q}x)x_i dqdp.$$

Since $\sum_i x_i(x_i, J_{p,q}x) = J_{p,q}x$ and $ve^{\theta J_{p,q}} = e^{\theta J_{p,q}}(\cos\theta \cdot v - \sin\theta \cdot J_{p,q}v)$ for $v \in V_N$, we have

$$[t_1, x] = \int\limits_{p\in S^{N-1}} \int\limits_{q\in S^{N-2}(p)} \int\limits_{-\pi}^{\pi} 2c(\theta)\sin\theta e^{\theta J_{p,q}} \big(\sin\theta \cdot x + (1+\cos\theta)\cdot J_{p,q}x\big) d\theta dqdp. \tag{5}$$

The right hand side of (5) can be written as $[x, C']$, where

$$C' := \int\limits_{p\in S^{N-1}} \int\limits_{q\in S^{N-2}(p)} \left(\int\limits_{-\pi}^{\pi} c(\theta)(-2-2\cos\theta)e^{\theta J_{p,q}} d\theta\right) dqdp.$$

Thus, it suffices to prove that $C' = C$.

The following has been established during the proof of Theorem 1.4:

$$\int\limits_{p\in S^{N-1}} \int\limits_{q\in S^{N-2}(p)} J_{p,q}^s dqdp = F_{s-1} = \mu_{s-1}\big[z^s\big]B(z), \tag{6}$$

$$\int\limits_{p\in S^{N-1}} \int\limits_{q\in S^{N-2}(p)} (x, J_{p,q}y)J_{p,q}^s dqdp = I_{s;x,y} = \nu_s\big[z^{s-1}\big]A(z)(x,y). \tag{7}$$

Let $c(\theta) = c_0\delta_0 + c_2\delta_0'' + c_4\delta_0^{(4)} + \ldots$ be the distribution from (1), where $\delta_0^{(k)}$ is the k-th derivative of the delta-function. Since

$$\int\limits_{-\pi}^{\pi} 2c(\theta)\sin\theta e^{\theta J_{p,q}} d\theta = 2\sum_{j\ge 1} c_j \sum_{l=1}^{\lfloor \frac{j+1}{2}\rfloor} (-1)^{l+1}\binom{j}{2l-1} J_{p,q}^{j-2l+1},$$

formulas (1) and (7) imply

$$[x, y] = \mathrm{Res}_{z=0}\, \bar{\zeta}\big(z^{-2}\big)A(z)(x,y)z^{-1}dz,$$

where $\bar{\zeta}(z^{-2}) = \sum_{j\ge 0} \bar{\zeta}_j z^{-2j}$ and $\bar{\zeta}_j = 2\nu_{2j+1}\sum_{l\ge 1}(-1)^{l+1}\binom{2j+2l}{2l-1}c_{2j+2l}$.

Comparing with $[x, y] = \mathrm{Res}_{z=0}\, \zeta(z^{-2})A(z)(x, y)z^{-1}dz$, we get $\bar{\zeta}(z^{-2}) = \zeta(z^{-2})$ and so $c_{2s+2} = a_s$, where $a_{>m} := 0$. On the other hand,

$$\int_{-\pi}^{\pi} c(\theta)(-2\cos\theta - 2)e^{\theta J_{p,q}}d\theta = 2\sum_{j\geq 0} c_j\left(-2J_{p,q}^{j} + \sum_{l=1}^{\lfloor j/2\rfloor}(-1)^{l+1}\binom{j}{2l}J_{p,q}^{j-2l}\right).$$

Combining this equality with (6), we find:

$$C' = \mathrm{Res}_{z=0}\, g\big(z^{-2}\big)B(z)z^{-1}dz = C.$$

This completes the proof of the theorem. □

Acknowledgements

The author is grateful to P. Etingof and I. Losev for numerous stimulating discussions. Special thanks are due to S. Sam for pointing out the result of Claim 2.3. Finally, the author is grateful to the referee for the useful comments on the final version of this paper.

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