SEVERAL REALIZATIONS OF FOCK MODULES FOR TOROIDAL $\ddot{U}_{q,d}(\mathfrak{sl}_n)$

ALEXANDER TSYMBALIUK

Abstract. In this paper, we relate the well-known *Fock representations* of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ to the vertex, shuffle, and 'L-operator' representations of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. These identifications generalize those for the quantum toroidal algebra of $\mathfrak{gl}_1$, which were established in [FJMM2].

Introduction

In the recent paper [FJMM2], authors proposed a shuffle approach to the Bethe ansatz problem for certain modules over the quantum toroidal algebra of $\mathfrak{gl}_1$ (*double* of the *small shuffle* algebra). The general idea behind a shuffle approach is that it frequently allows to interpret complicated concepts in simple terms. As the theory of quantum toroidal algebras of $\mathfrak{sl}_n$ is very similar to that for quantum toroidal algebras of $\mathfrak{gl}_1$ (though technically it is more involved), it is desirable to generalize the aforementioned construction for the former case.

In this article, we identify different families of representations of quantum toroidal algebras of $\mathfrak{sl}_n$. This will be crucial for our arguments in [FT2], where we diagonalize the commutative subalgebras of the quantum toroidal algebras of $\mathfrak{sl}_n$ studied in [FT1].

This paper is organized as follows:

• In Section 1, we recall the definition and key results about the quantum toroidal algebra $\ddot{U}_{q,d}(\mathfrak{sl}_n), n > 2$. In particular, we recall the relation to the shuffle algebra S (of $A^{(1)}_{n-1}$-type) studied in [N, FT1]. We also remind three different constructions of their representations:

-combinatorial representations $\tau^p_{u,\bar{c}}$ introduced in [FJMM1],

-vertex representations $\rho^p_{u,\bar{c}}$ constructed in [S],

-shuffle representations $\pi^p_{u,\bar{c}}$ which we define following [FJMM2].

• In Section 2, we relate the aforementioned three different families of representations:

-In Theorem 2.2, we show that $\pi^p_{u,\bar{c}}$ induces an action on the factor of $S_{1,p}(u)$ by the right S'-action (here $S' \subset S$ denotes the augmentation ideal), which is isomorphic to the $\tau^p_{u,\bar{c}}$-action. In Theorems 2.5, 2.6, we generalize this result to some other families of representations.

-In Theorem 2.7, we show that Miki's isomorphism ϖ of the quantum toroidal algebras intertwines the dual of the combinatorial representation $\tau^p_{u,\bar{c}}$ and the corresponding vertex representations $\rho^p_{u',\bar{c}'}$ for appropriate parameters.

• In Section 3, we study the matrix elements of L operators associated to the vertex representations $\rho^p_{u,\bar{c}}$. In Theorem 3.5, we derive an explicit formula for the matrix element $L^{p,\bar{c}}_{\emptyset,\emptyset}$, whose shuffle realization was obtained in [FT1]. This allows us to identify the shuffle S-bimodule $S_{1,p}(u)$ with the S-bimodule generated by $L^{p,\bar{c}}_{\emptyset,\emptyset}$.

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1. Basic definitions and constructions

1.1. Quantum toroidal algebras of $\mathfrak{sl}_n$ for $n \geq 3$.

Let $q, d \in \mathbb{C}^\times$ be two parameters. We set $[n] := \{0, 1, \ldots, n-1\}$, $[n]^\times := [n]\backslash\{0\}$, the former viewed as a set of mod n residues. Let $g_m(z) := \frac{q^m z - 1}{z - q^m}$. Define $\{a_{i,j}, m_{i,j} | i, j \in [n]\}$ by

$$a_{i,i} = 2,\ a_{i,i\pm 1} = -1,\ m_{i,i\pm 1} = \mp 1,\ \text{and } a_{i,j} = m_{i,j} = 0 \text{ otherwise.}$$

The quantum toroidal algebra of $\mathfrak{sl}_n$, denoted by $\ddot{U}_{q,d}(\mathfrak{sl}_n)$, is the unital associative $\mathbb{C}$-algebra generated by $\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi_{i,0}^{-1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{i \in [n]}^{k \in \mathbb{Z}}$ with the following defining relations:

$$\text{(T0.1)} \qquad [\psi_i^\pm(z), \psi_j^\pm(w)] = 0,\ \gamma^{\pm 1/2} - \text{central},$$

$$\text{(T0.2)} \qquad \psi_{i,0}^{\pm 1} \cdot \psi_{i,0}^{\mp 1} = \gamma^{\pm 1/2} \cdot \gamma^{\mp 1/2} = q^{\pm d_1} \cdot q^{\mp d_1} = q^{\pm d_2} \cdot q^{\mp d_2} = 1,$$

$$\text{(T0.3)} \qquad q^{d_1} e_i(z) q^{-d_1} = e_i(qz),\ q^{d_1} f_i(z) q^{-d_1} = f_i(qz),\ q^{d_1} \psi_i^\pm(z) q^{-d_1} = \psi_i^\pm(qz),$$

$$\text{(T0.4)} \qquad q^{d_2} e_i(z) q^{-d_2} = q e_i(z),\ q^{d_2} f_i(z) q^{-d_2} = q^{-1} f_i(z),\ q^{d_2} \psi_i^\pm(z) q^{-d_2} = \psi_i^\pm(z),$$

$$\text{(T1)} \qquad g_{a_{i,j}}(\gamma^{-1} d^{m_{i,j}} z/w) \psi_i^+(z) \psi_j^-(w) = g_{a_{i,j}}(\gamma d^{m_{i,j}} z/w) \psi_j^-(w) \psi_i^+(z),$$

$$\text{(T2)} \qquad e_i(z) e_j(w) = g_{a_{i,j}}(d^{m_{i,j}} z/w) e_j(w) e_i(z),$$

$$\text{(T3)} \qquad f_i(z) f_j(w) = g_{a_{i,j}}(d^{m_{i,j}} z/w)^{-1} f_j(w) f_i(z),$$

$$\text{(T4)} \qquad (q - q^{-1})[e_i(z), f_j(w)] = \delta_{i,j} \left(\delta(\gamma w/z) \psi_i^+(\gamma^{1/2} w) - \delta(\gamma z/w) \psi_i^-(\gamma^{1/2} z) \right),$$

$$\text{(T5)} \qquad \psi_i^\pm(z) e_j(w) = g_{a_{i,j}}(\gamma^{\pm 1/2} d^{m_{i,j}} z/w) e_j(w) \psi_i^\pm(z),$$

$$\text{(T6)} \qquad \psi_i^\pm(z) f_j(w) = g_{a_{i,j}}(\gamma^{\mp 1/2} d^{m_{i,j}} z/w)^{-1} f_j(w) \psi_i^\pm(z),$$

$$\text{(T7.1)} \qquad \mathrm{Sym}_{z_1, z_2}\ [e_i(z_1), [e_i(z_2), e_{i\pm 1}(w)]_q]_{q^{-1}} = 0,\ [e_i(z), e_j(w)] = 0 \text{ for } j \neq i, i \pm 1,$$

$$\text{(T7.2)} \qquad \mathrm{Sym}_{z_1, z_2}\ [f_i(z_1), [f_i(z_2), f_{i\pm 1}(w)]_q]_{q^{-1}} = 0,\ [f_i(z), f_j(w)] = 0 \text{ for } j \neq i, i \pm 1,$$

where we set $[a, b]_x := ab - x \cdot ba$ and define the generating series as follows:

$$e_i(z) := \sum_{k=-\infty}^{\infty} e_{i,k} z^{-k},\ f_i(z) := \sum_{k=-\infty}^{\infty} f_{i,k} z^{-k},\ \psi_i^\pm(z) := \psi_{i,0}^{\pm 1} + \sum_{r>0} \psi_{i,\pm r} z^{\mp r},\ \delta(z) := \sum_{k=-\infty}^{\infty} z^k.$$

It will be convenient to use the generators $\{h_{i,k}\}_{k \neq 0}$ instead of $\{\psi_{i,k}\}_{k \neq 0}$, defined by

$$\exp\left(\pm(q - q^{-1}) \sum_{r>0} h_{i,\pm r} z^{\mp r} \right) = \bar{\psi}_i^\pm(z) := \psi_{i,0}^{\mp 1} \psi_i^\pm(z),\quad h_{i,\pm r} \in \mathbb{C}[\psi_{i,0}^{\mp 1}, \psi_{i,\pm 1}, \psi_{i,\pm 2}, \ldots].$$

Then the relations (T5,T6) are equivalent to the following (we use $[m]_q := (q^m - q^{-m})/(q - q^{-1})$):

$$\text{(T5}'\text{)} \qquad \psi_{i,0} e_{j,l} = q^{a_{i,j}} e_{j,l} \psi_{i,0},\ [h_{i,k}, e_{j,l}] = d^{-k m_{i,j}} \gamma^{-|k|/2} \frac{[k a_{i,j}]_q}{k} e_{j,l+k}\ (k \neq 0),$$

$$\text{(T6}'\text{)} \qquad \psi_{i,0} f_{j,l} = q^{-a_{i,j}} f_{j,l} \psi_{i,0},\ [h_{i,k}, f_{j,l}] = -d^{-k m_{i,j}} \gamma^{|k|/2} \frac{[k a_{i,j}]_q}{k} f_{j,l+k}\ (k \neq 0).$$

We also introduce $h_{i,0}, c, c'$ via $\psi_{i,0} = q^{h_{i,0}},\ \gamma^{1/2} = q^c,\ c' = \sum_{i \in [n]} h_{i,0}$, so that c, c' are central.

For our discussion in Sections 1.3–1.4, we will also need to make sense of the elements $q^{\frac{h_{i,0}}{2n}}, \gamma^{\frac{1}{2n}}, q^{\frac{d_2}{n}}$. In such cases, we formally add elements of the form $q^{\frac{h_{i,0}}{N}}, q^{\frac{c}{N}}, q^{\frac{d_1}{N}}, q^{\frac{d_2}{N}}\ \forall\, N \in \mathbb{N}$.

1.2. **Hopf algebra structure, Hopf pairing, and Drinfeld double.**

In this section, we recall some of the basic results on $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ which are relevant for us.

• *Hopf algebra structure.*

Following [DI, Theorem 2.1], we endow the quantum toroidal algebra $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ with a topological Hopf algebra structure by assigning (we use $\gamma_{(1)} := \gamma \otimes 1$ and $\gamma_{(2)} := 1 \otimes \gamma$)

(H1) $\Delta(\psi_i^{\pm}(z)) = \psi_i^{\pm}(\gamma_{(2)}^{\pm 1/2} z) \otimes \psi_i^{\pm}(\gamma_{(1)}^{\mp 1/2} z),\ \Delta(x) = x \otimes x$ for $x = \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}$,

$$\Delta(e_i(z)) = e_i(z) \otimes 1 + \psi_i^-(\gamma_{(1)}^{1/2} z) \otimes e_i(\gamma_{(1)} z),\ \Delta(f_i(z)) = 1 \otimes f_i(z) + f_i(\gamma_{(2)} z) \otimes \psi_i^+(\gamma_{(2)}^{1/2} z),$$

(H2) $\epsilon(e_i(z)) = \epsilon(f_i(z)) = 0,\ \epsilon(\psi_i^{\pm}(z)) = 1,\ \epsilon(x) = 1$ for $x = \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}$,

(H3) $S(e_i(z)) = -\psi_i^-(\gamma^{-1/2} z)^{-1} e_i(\gamma^{-1} z),\ S(f_i(z)) = -f_i(\gamma^{-1} z) \psi_i^+(\gamma^{-1/2} z)^{-1},$

$$S(x) = x^{-1} \text{ for } x = \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}, \psi_i^{\pm}(z).$$

• *Sub/quotient-algebras of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.*

In what follows, we will need the following subalgebras of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$:

∘ $\ddot{U}^{\geq}$ is the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,k}, \psi_{i,l}, \psi_{i,0}^{\pm 1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{i \in [n]}^{k \in \mathbb{Z}, l \in -\mathbb{N}}$.

∘ $\ddot{U}^{\leq}$ is the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{f_{i,k}, \psi_{i,l}, \psi_{i,0}^{\pm 1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{i \in [n]}^{k \in \mathbb{Z}, l \in \mathbb{N}}$.

∘ $\ddot{U}^{+}, \ddot{U}^{-}$ are the subalgebras of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,k}\}_{i \in [n]}^{k \in \mathbb{Z}}$ and $\{f_{i,k}\}_{i \in [n]}^{k \in \mathbb{Z}}$, respectively.

∘ $\ddot{U}^{0}$ is the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{\psi_{i,k}, \psi_{i,0}^{-1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{i \in [n]}^{k \in \mathbb{Z}}$.

We also define two modifications of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$:

∘ Let $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ be obtained from $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ by "ignoring" the generator $q^{\pm d_2}$ and taking quotient by the ideal (c'), i.e., setting $c' = 0$. It is clear how to define the subalgebras $\ddot{U}'^{\geq}, \ddot{U}'^{\leq}, \ddot{U}'^{\pm}, \ddot{U}'^{0}$.

∘ Let $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ be obtained from $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ by "ignoring" the generator $q^{\pm d_1}$ and taking quotient by the ideal (c), i.e., setting $c = 0$. It is clear how to define the subalgebras $'\ddot{U}^{\geq}, '\ddot{U}^{\leq}, '\ddot{U}^{\pm}, '\ddot{U}^{0}$.

• *Hopf pairing and Drinfeld double.*

Analogously to the case of quantum affine algebras, we have the following result.

Theorem 1.1. *(a) There exists a unique Hopf algebra pairing $\varphi : \ddot{U}^{\geq} \times \ddot{U}^{\leq} \to \mathbb{C}$ satisfying*

$$\varphi(e_i(z), f_j(w)) = \frac{\delta_{i,j}}{q - q^{-1}} \cdot \delta\left(\frac{z}{w}\right),\ \varphi(\psi_i^-(z), \psi_j^+(w)) = g_{a_{i,j}}(d^{m_{i,j}} z/w),\ \varphi(q^{d_2}, q^{d_2}) = q^{\frac{n(n^2-1)}{12}},$$

$$\varphi(e_i(z), x^-) = \varphi(x^+, f_i(z)) = 0 \text{ for } x^{\pm} = \psi_j^{\mp}(w), \gamma^{1/2}, q^{d_1}, q^{d_2},$$

$$\varphi(\gamma^{1/2}, q^{d_1}) = \varphi(q^{d_1}, \gamma^{1/2}) = q^{-1/2},\ \varphi(\psi_i^-(z), q^{d_2}) = q^{-1},\ \varphi(q^{d_2}, \psi_i^+(z)) = q,$$

$$\varphi(\psi_i^-(z), x) = \varphi(x, \psi_i^+(z)) = 1 \text{ for } x = \gamma^{1/2}, q^{d_1},$$

$$\varphi(\gamma^{1/2}, q^{d_2}) = \varphi(q^{d_2}, \gamma^{1/2}) = \varphi(\gamma^{1/2}, \gamma^{1/2}) = \varphi(q^{d_1}, q^{d_1}) = \varphi(q^{d_1}, q^{d_2}) = \varphi(q^{d_2}, q^{d_1}) = 1.$$

(b) The natural Hopf algebra homomorphism from the Drinfeld double $D_\varphi(\ddot{U}^{\geq}, \ddot{U}^{\leq})$ to $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ induces the isomorphism

$$\Xi : D_\varphi(\ddot{U}^{\geq}, \ddot{U}^{\leq})/I \xrightarrow{\sim} \ddot{U}_{q,d}(\mathfrak{sl}_n) \text{ with } I := (x \otimes 1 - 1 \otimes x | x = \psi_{i,0}^{\pm 1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}).$$

(c) Analogously to (b), the algebras $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ and $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ admit the Drinfeld double realizations via $D_{\varphi'}(\ddot{U}'^{\geq}, \ddot{U}'^{\leq})$ and $D_{'\varphi}('\ddot{U}^{\geq}, '\ddot{U}^{\leq})$, where φ' and $'\varphi$ are defined similarly to φ.

(d) The pairings $\varphi, \varphi', '\varphi$ are nondegenerate if and only if q, qd, qd^{-1} are not roots of unity.

(e) If q, qd, qd^{-1} are not roots of unity, then the algebras $\ddot{U}_{q,d}(\mathfrak{sl}_n), \ddot{U}'_{q,d}(\mathfrak{sl}_n)$ and $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ admit the universal R-matrices R, R' and $'R$, associated to the pairings φ, φ' and $'\varphi$, respectively.

1.3. **Two copies of $U_q(\widehat{\mathfrak{sl}}_n)$ inside $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.**

Let $U_q(\widehat{\mathfrak{sl}}_n)$ be the quantum affine algebra of $\mathfrak{sl}_n$ presented in the *new Drinfeld realization.* This is the unital associative $\mathbb{C}$-algebra generated by $\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi_{i,0}^{-1}, C^{\pm 1}, \widetilde{D}^{\pm 1}\}_{i\in[n]^\times}^{k\in\mathbb{Z}}$ with the defining relations similar to those of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ (see [M2]):

(A0.1) $$[\psi_i^\pm(z), \psi_j^\pm(w)] = 0,\ C^{\pm 1} - \text{central},$$

(A0.2) $$\psi_{i,0}^{\pm 1}\cdot\psi_{i,0}^{\mp 1} = C^{\pm 1}\cdot C^{\mp 1} = \widetilde{D}^{\pm 1}\cdot\widetilde{D}^{\mp 1} = 1,$$

(A0.3) $$\widetilde{D}e_i(z)\widetilde{D}^{-1} = qe_i(q^{-n}z),\ \widetilde{D}f_i(z)\widetilde{D}^{-1} = q^{-1}f_i(q^{-n}z),\ \widetilde{D}\psi_i^\pm(z)\widetilde{D}^{-1} = \psi_i^\pm(q^{-n}z),$$

(A1) $$g_{a_{i,j}}(C^{-1}z/w)\psi_i^+(z)\psi_j^-(w) = \psi_j^-(w)\psi_i^+(z)g_{a_{i,j}}(Cz/w),$$

(A2) $$e_i(z)e_j(w) = g_{a_{i,j}}(z/w)e_j(w)e_i(z),$$

(A3) $$f_i(z)f_j(w) = g_{a_{i,j}}(z/w)^{-1}f_j(w)f_i(z),$$

(A4) $$(q-q^{-1})[e_i(z), f_j(w)] = \delta_{i,j}\left(\delta(Cw/z)\psi_i^+(Cw) - \delta(Cz/w)\psi_i^-(Cz)\right),$$

(A5) $$\psi_i^+(z)e_j(w) = g_{a_{i,j}}(z/w)e_j(w)\psi_i^+(z),\ \psi_i^-(z)e_j(w) = g_{a_{i,j}}(C^{-1}z/w)e_j(w)\psi_i^-(z),$$

(A6) $$\psi_i^+(z)f_j(w) = g_{a_{i,j}}(C^{-1}z/w)^{-1}f_j(w)\psi_i^+(z),\ \psi_i^-(z)f_j(w) = g_{a_{i,j}}(z/w)^{-1}f_j(w)\psi_i^-(z),$$

(A7.1) $$\mathrm{Sym}_{z_1,z_2}\,[e_i(z_1),[e_i(z_2),e_j(w)]_q]_{q^{-1}} = 0 \text{ if } a_{i,j}=-1,\ [e_i(z),e_j(w)] = 0 \text{ if } a_{i,j}=0,$$

(A7.2) $$\mathrm{Sym}_{z_1,z_2}\,[f_i(z_1),[f_i(z_2),f_j(w)]_q]_{q^{-1}} = 0 \text{ if } a_{i,j}=-1,\ [f_i(z),f_j(w)] = 0 \text{ if } a_{i,j}=0,$$

where the generating series $e_i(z), f_i(z), \psi_i^\pm(z)$ are defined as before.

This algebra is known to admit a classical Drinfeld–Jimbo realization. In other words, it is generated by $\{x_i^\pm, t_i^{\pm 1}, D^{\pm 1}\}_{i\in[n]}$ with the standard defining relations:

$$D^{\pm 1}D^{\mp 1} = 1,\ Dt_iD^{-1} = t_i,\ Dx_i^\pm D^{-1} = q^{\pm 1}x_i^\pm,$$
$$t_i^{\pm 1}t_i^{\mp 1} = 1,\ t_it_j = t_jt_i,\ t_ix_j^\pm t_i^{-1} = q^{\pm a_{i,j}}x_j^\pm,$$
$$[x_i^+, x_j^-] = \delta_{i,j}\cdot\frac{t_i - t_i^{-1}}{q-q^{-1}},\ \sum_{s=0}^{1-a_{i,j}}\frac{(-1)^s}{[s]_q![1-a_{i,j}-s]_q!}(x_i^\pm)^s x_j^\pm (x_i^\pm)^{1-a_{i,j}-s} = 0.$$

An explicit identification of these two presentations of $U_q(\widehat{\mathfrak{sl}}_n)$ is given by

$$x_i^+ = e_{i,0},\ x_i^- = f_{i,0},\ t_i^{\pm 1} = \psi_{i,0}^{\pm 1}\ (1\le i\le n-1),\ t_0 = C\cdot(\psi_{1,0}\cdots\psi_{n-1,0})^{-1},\ D = \widetilde{D},$$
$$x_0^+ = C(\psi_{1,0}\cdots\psi_{n-1,0})^{-1}[\cdots[f_{1,1}, f_{2,0}]_q, \cdots, f_{n-1,0}]_q,$$
$$x_0^- = [e_{n-1,0}, \cdots, [e_{2,0}, e_{1,-1}]_{q^{-1}}\cdots]_{q^{-1}}(\psi_{1,0}\cdots\psi_{n-1,0})C^{-1}.$$

Following [VV], we introduce the *vertical* and *horizontal* copies of $U_q(\widehat{\mathfrak{sl}}_n)$ inside $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. Consider two algebra homomorphisms $h, v : U_q(\widehat{\mathfrak{sl}}_n) \to \ddot{U}_{q,d}(\mathfrak{sl}_n)$ defined by

$$h : x_i^+ \mapsto e_{i,0},\ x_i^- \mapsto f_{i,0},\ t_i \mapsto \psi_{i,0},\ D \mapsto q^{d_2},$$

$$v : e_{i,k} \mapsto d^{ik}e_{i,k},\ f_{i,k} \mapsto d^{ik}f_{i,k},\ \psi_{i,k} \mapsto d^{ik}\gamma^{k/2}\psi_{i,k},\ C \mapsto \gamma,\ \widetilde{D} \mapsto q^{-nd_1}\cdot q^{\sum_{j=1}^{n-1}\frac{j(n-j)}{2}h_{j,0}},$$

where we follow the convention from Section 1.1 and 'add' elements $q^{h_{j,0}/2}$ to $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

It is known that both h, v are inclusions. The images of h and v, denoted by $\dot{U}_q^{\mathrm{h}}(\mathfrak{sl}_n)$ and $\dot{U}_q^{\mathrm{v}}(\mathfrak{sl}_n)$, are called the *horizontal* and *vertical* copies of $U_q(\widehat{\mathfrak{sl}}_n)$ inside $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

1.4. **Miki's isomorphism.**

We recall the beautiful result of Miki which provides an isomorphism $'\ddot{U}_{q,d}(\mathfrak{sl}_n)\xrightarrow{\sim}\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ intertwining the *vertical* and *horizontal* embeddings of quantum affine algebras of $\mathfrak{sl}_n$.

To formulate the main result of this section, we need some more notation.

$\circ$ Let $U_q(L\mathfrak{sl}_n)$ be obtained from $U_q(\widehat{\mathfrak{sl}}_n)$ by "ignoring" the generator $\widetilde{D}^{\pm 1}$ and taking quotient by the ideal $(C-1)$, i.e., setting $C=1$. The algebra $U_q(L\mathfrak{sl}_n)$ is usually called the quantum loop algebra of $\mathfrak{sl}_n$. Analogously to h and v, we have the following monomorphisms:

$$'\ddot{U}_{q,d}(\mathfrak{sl}_n) \overset{'h}{\hookleftarrow} U_q(\widehat{\mathfrak{sl}}_n) \overset{v'}{\hookrightarrow} \ddot{U}'_{q,d}(\mathfrak{sl}_n)$$

and

$$'\ddot{U}_{q,d}(\mathfrak{sl}_n) \overset{'v}{\hookleftarrow} U_q(L\mathfrak{sl}_n) \overset{h'}{\hookrightarrow} \ddot{U}'_{q,d}(\mathfrak{sl}_n).$$

$\circ$ Let σ be the antiautomorphism of $U_q(\widehat{\mathfrak{sl}}_n)$ determined by

$$\sigma: x_i^{\pm}\mapsto x_i^{\pm},\ t_i\mapsto t_i^{-1},\ D\mapsto D^{-1}.$$

$\circ$ Let η be the antiautomorphism of $U_q(\widehat{\mathfrak{sl}}_n)$ determined by

$$\eta: e_{i,k}\mapsto e_{i,-k},\ f_{i,k}\mapsto f_{i,-k},\ h_{i,l}\mapsto -C^l h_{i,-l},\ \psi_{i,0}\mapsto \psi_{i,0}^{-1},\ C\mapsto C,\ \widetilde{D}\mapsto \widetilde{D}\cdot\prod_{i=1}^{n-1}\psi_{i,0}^{-i(n-i)}.$$

$\circ$ Let $'Q$ be the automorphism of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ determined by

$$'Q: e_{i,k}\mapsto (-d)^k e_{i+1,k},\ f_{i,k}\mapsto (-d)^k f_{i+1,k},\ h_{i,l}\mapsto (-d)^l h_{i+1,l},\ \psi_{i,0}\mapsto \psi_{i+1,0},\ q^{d_2}\mapsto q^{d_2}.$$

$\circ$ Let Q' be the automorphism of $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ such that it maps the generators other than $\gamma^{\pm 1/2}, q^{\pm d_1}$ as $'Q$, while

$$Q': \gamma^{\pm 1/2}\mapsto\gamma^{\pm 1/2},\ q^{\pm d_1}=q^{\pm d_1}\cdot\gamma^{\mp 1}.$$

$\circ$ Let $'\mathcal{Y}_j$ $(1\le j\le n)$ be the automorphism of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ determined by

$$'\mathcal{Y}_j: h_{i,l}\mapsto h_{i,l},\ \psi_{i,0}^{\pm 1}\mapsto\psi_{i,0}^{\pm 1},\ q^{\pm d_2}\mapsto q^{\pm d_2},$$

$$e_{i,k}\mapsto (-d)^{-n\delta_{i,0}\delta_{j,n}-i\bar{\delta}_{i,j}+i\delta_{i,j-1}} e_{i,k-\bar{\delta}_{i,j}+\delta_{i,j-1}},\ f_{i,k}\mapsto (-d)^{n\delta_{i,0}\delta_{j,n}+i\bar{\delta}_{i,j}-i\delta_{i,j-1}} f_{i,k+\bar{\delta}_{i,j}-\delta_{i,j-1}},$$

where $\bar{\delta}_{i,j}=\begin{cases}1 & \text{if } j\equiv i \pmod n\\ 0 & \text{otherwise}\end{cases}$.

$\circ$ Let $\mathcal{Y}'_j$ $(1\le j\le n)$ be the automorphism of $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ such that it maps the generators other than $\psi_{i,0}^{\pm 1},\gamma^{\pm 1/2},q^{\pm d_1}$ as $'\mathcal{Y}_j$, while

$$\mathcal{Y}'_j: \gamma^{\pm 1/2}\mapsto\gamma^{\pm 1/2},\ \psi_{i,0}^{\pm 1}\mapsto\gamma^{\mp\bar{\delta}_{i,j}\pm\delta_{i,j-1}}\psi_{i,0}^{\pm 1},$$

$$\mathcal{Y}'_j: q^{\pm d_1}\mapsto q^{\pm d_1}\cdot\gamma^{\mp\frac{n+1}{2n}}\cdot K_j^{\pm 1} \text{ with } K_j=\prod_{l=1}^{j-1} q^{\frac{l}{n}h_{l,0}}\prod_{l=j}^{n-1} q^{\frac{l-n}{n}h_{l,0}},$$

where we follow the convention from Section 1.1 and 'add' elements $\gamma^{\frac{1}{2n}}, q^{\frac{h_{j,0}}{2n}}$ to $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$.

Theorem 1.2. [M2, Proposition 1] *There exists an algebra isomorphism*

$$\varpi:\ '\ddot{U}_{q,d}(\mathfrak{sl}_n)\xrightarrow{\sim}\ddot{U}'_{q,d}(\mathfrak{sl}_n)$$

satisfying the following properties:

$$\varpi\circ{}'h=v',\ \varpi\circ{}'v\circ\eta\circ\sigma=h',\ Q'\circ\mathcal{Y}'_n\circ\varpi=\varpi\circ{}'\mathcal{Y}_1^{-1}\circ{}'Q.$$

Remark 1.3. *(i) Let $U^{\mathrm{tor}}_{q,d}(\mathfrak{sl}_n)$ be obtained from $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ by "ignoring" the generators $q^{\pm d_1}$ and $q^{\pm d_2}$. First Miki established (see* [M1]*) an isomorphism $\varpi: U^{\mathrm{tor}}_{q,d}(\mathfrak{sl}_n)\xrightarrow{\sim}U^{\mathrm{tor}}_{q,d}(\mathfrak{sl}_n)$.*
(ii) The generators $x^{\pm}_{i,k}, h_{i,l}, k_i^{\pm 1}, C^{\pm 1}, D^{\pm 1}, \widetilde{D}^{\pm 1}$ from [M2] *are related to our generators via*

$$x^+_{i,k}\leftrightarrow d^{ik}e_{i,k},\ x^-_{i,k}\leftrightarrow d^{ik}f_{i,k},\ h_{i,l}\leftrightarrow d^{il}\gamma^{l/2}h_{i,l},$$

$$k_i^{\pm 1}\leftrightarrow \psi^{\pm 1}_{i,0},\ C^{\pm 1}\leftrightarrow \gamma^{\pm 1},\ D^{\pm 1}\leftrightarrow q^{\pm d_2},\ \widetilde{D}^{\pm 1}\leftrightarrow q^{\mp nd_1}\cdot q^{\pm\sum_{j=1}^{n-1}\frac{j(n-j)}{2}h_{j,0}},$$

while the parameters q,ξ from [M2] *are related to our parameters q,d via*

$$q\leftrightarrow q,\ \xi\leftrightarrow d^{-n}.$$

(iii) The aforementioned choice of generators from [M2] *is convenient as there is no need to adjoint elements of the form $q^{\frac{h_{j,0}}{N}}, q^{\frac{c}{N}}, q^{\frac{d_1}{N}}, q^{\frac{d_2}{N}}$. However, our choice of generators is more symmetric and suitable for the rest of exposition.*

We conclude this section by computing images of some generators under ϖ.

Proposition 1.4. *(a) We have*

$$\varpi: e_{i,0}\mapsto e_{i,0},\ f_{i,0}\mapsto f_{i,0},\ \psi^{\pm 1}_{i,0}\mapsto \psi^{\pm 1}_{i,0}\ \text{for}\ i\in[n]^\times,$$

$$\varpi: \psi^{\pm 1}_{0,0}\mapsto \gamma^{\pm 1}\cdot\psi^{\pm 1}_{0,0},\ q^{\pm d_2}\mapsto q^{\mp nd_1}\cdot q^{\pm\sum_{j=1}^{n-1}\frac{j(n-j)}{2}h_{j,0}},$$

$$\varpi: e_{0,0}\mapsto d\cdot\gamma\psi_{0,0}\cdot[\cdots[f_{1,1},f_{2,0}]_q,\cdots,f_{n-1,0}]_q,$$

$$\varpi: f_{0,0}\mapsto d^{-1}\cdot[e_{n-1,0},\cdots,[e_{2,0},e_{1,-1}]_{q^{-1}}\cdots]_{q^{-1}}\cdot\psi^{-1}_{0,0}\gamma^{-1}.$$

(b) For $i\in[n]^\times$, we have

$$\varpi(h_{i,1})=(-1)^{i+1}d^{-i}\cdot[[\cdots[[\cdots[f_{0,0},f_{n-1,0}]_q,\cdots,f_{i+1,0}]_q,f_{1,0}]_q,\cdots,f_{i-1,0}]_q,f_{i,0}]_{q^2},$$

$$\varpi(h_{i,-1})=(-1)^{i+1}d^{i}\cdot[e_{i,0},[\cdots,[e_{1,0},[e_{i+1,0},\cdots,[e_{n-1,0},e_{0,0}]_{q^{-1}}\cdots]_{q^{-1}}]_{q^{-1}}\cdots]_{q^{-1}}]_{q^{-2}}.$$

(c) For $i=0$, we have

$$\varpi(h_{0,1})=(-1)^n d^{1-n}\cdot[[\cdots[f_{1,1},f_{2,0}]_q,\cdots,f_{n-1,0}]_q,f_{0,-1}]_{q^2},$$

$$\varpi(h_{0,-1})=(-1)^n d^{n-1}\cdot[e_{0,1},[e_{n-1,0},\cdots,[e_{2,0},e_{1,-1}]_{q^{-1}}\cdots]_{q^{-1}}]_{q^{-2}}.$$

(d) We have

$$\varpi(e_{0,-1})=(-d)^n e_{0,1},\ \varpi(f_{0,1})=(-d)^{-n}f_{0,-1}.$$

Proof.
(a) Follows straightforwardly from the equality $\varpi\circ{}'h=v'$ and the aforementioned identification of the Drinfeld–Jimbo and the new Drinfeld presentations of $U_q(\widehat{\mathfrak{sl}}_n)$.
(b) Applying iteratively the following simple identity (see ($\diamondsuit$) from Section 3)

$$[a,[b,c]_u]_v=[[a,b]_x,c]_{uv/x}+x\cdot[b,[a,c]_{v/x}]_{u/x}\ (\forall\ u,v,x\in\mathbb{C}^\times)$$

to the formulas expressing $x_0^{\pm}\in U_q(\widehat{\mathfrak{sl}}_n)$ in the new Drinfeld realization, one can easily deduce the formulas expressing $h_{i,\pm 1}\in U_q(\widehat{\mathfrak{sl}}_n)$ $(i\in[n]^\times)$ in the Drinfeld–Jimbo realization

$$h_{i,1}=(-1)^i[[\cdots[[\cdots[x_0^+,x_{n-1}^+]_{q^{-1}},\cdots,x_{i+1}^+]_{q^{-1}},x_1^+]_{q^{-1}},\cdots,x_{i-1}^+]_{q^{-1}},x_i^+]_{q^{-2}},$$

$$h_{i,-1}=(-1)^i[x_i^-,\cdots,[x_1^-,[x_{i+1}^-,\cdots,[x_{n-1}^-,x_0^-]_q\cdots]_q]_q\cdots]_{q^2}.$$

It remains to use the equality $\varpi\circ{}'v\circ\eta=h'\circ\sigma^{-1}$.
(c) Formulas for $\varpi(h_{0,\pm 1})$ follow from the formulas for $\varpi(h_{n-1,\pm 1})$ from (b) by applying the equality $Q'\circ\mathcal{Y}_n'\circ\varpi=\varpi\circ{}'\mathcal{Y}_1^{-1}\circ{}'Q$ to $h_{n-1,\pm 1}$.
(d) It suffices to apply the equality $Q'\circ\mathcal{Y}_n'\circ\varpi=\varpi\circ{}'\mathcal{Y}_1^{-1}\circ{}'Q$ to $e_{n-1,0}$ and $f_{n-1,0}$. $\square$

1.5. **Fock and Macmahon modules.**

In this section, we recall two interesting classes of ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-modules constructed in [FJMM1]. They depend on two parameters: $0 \le p \le n-1$ and $u \in \mathbb{C}^\times$. We also set

$$(\diamond) \qquad q_1 := q^{-1}d,\ q_2 := q^2,\ q_3 := q^{-1}d^{-1} \text{ and } \phi(t) := \frac{q^{-1}t - q}{t-1}.$$

Given a collection of formal series $\boldsymbol{\phi}(z) = \{\phi_i^\pm(z)\}_{i\in[n]},\ \phi_i^\pm(z) \in \mathbb{C}[[z^{\mp 1}]]$, a vector v of an ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-module V is said to have *weight* $\boldsymbol{\phi}(z)$ if $\psi_i^\pm(z)v = \phi_i^\pm(z)\cdot v\ \forall\ i \in [n]$. We say that V is a *lowest weight module* if it is generated by a weight vector v such that $f_i(z)v = 0\ \forall\ i \in [n]$. Such a v is called a *lowest weight vector*, and its weight the *lowest weight* of V. Given $\boldsymbol{\phi}(z)$ with $\phi_i^+(\infty)\phi_i^-(0) = 1$, there is a unique irreducible lowest weight module of that lowest weight. If $\phi_i^\pm(z)$ are expansions of a rational function $\phi_i(z)$ at $z = 0, \infty$, then we write $\boldsymbol{\phi}(z) = (\phi_i(z))_{i\in[n]}$.

• *Fock modules* $F^{(p)}(u)$.

The most basic lowest weight ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-module is the Fock module $F^{(p)}(u)$. It is a unique irreducible lowest weight module of the lowest weight $(\phi(z/u)^{\delta_{i,p}})_{i\in[n]}$.

Proposition 1.5. [FJMM1, Proposition 3.3] *As a vector space, $F^{(p)}(u)$ has a basis $\{|\lambda\rangle\}$ labeled by all partitions. In this basis, the ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action is given by the following formulas:*

$$\langle\lambda+1_j|e_i(z)|\lambda\rangle = \bar{\delta}_{p+j-\lambda_j,i+1} \prod_{1\le s<j}^{p+s-\lambda_s\equiv i} \phi(q_1^{\lambda_s-\lambda_j-1}q_3^{s-j}) \prod_{1\le s<j}^{p+s-\lambda_s\equiv i+1} \phi(q_1^{\lambda_j-\lambda_s}q_3^{j-s})\delta(q_1^{\lambda_j}q_3^{j-1}u/z),$$

$$\langle\lambda|f_i(z)|\lambda+1_j\rangle = \bar{\delta}_{p+j-\lambda_j,i+1} \prod_{s>j}^{p+s-\lambda_s\equiv i} \phi(q_1^{\lambda_s-\lambda_j-1}q_3^{s-j}) \prod_{s>j}^{p+s-\lambda_s\equiv i+1} \phi(q_1^{\lambda_j-\lambda_s}q_3^{j-s})\delta(q_1^{\lambda_j}q_3^{j-1}u/z),$$

$$\langle\lambda|\psi_i^\pm(z)|\lambda\rangle = \prod_{s\ge 1}^{p+s-\lambda_s\equiv i} \phi(q_1^{\lambda_s-1}q_3^{s-1}u/z) \prod_{s\ge 1}^{p+s-\lambda_s\equiv i+1} \phi(q_1^{\lambda_s-1}q_3^{s-2}u/z)^{-1},\ \langle\lambda|q^{d_2}|\lambda\rangle = q^{|\lambda|},$$

while all other matrix coefficients are zero. The vector $|\emptyset\rangle$ is the lowest weight vector of $F^{(p)}(u)$.

Definition 1.6. *For $\bar{c} \in (\mathbb{C}^\times)^{[n]}$, let $\tau^p_{u,\bar{c}}$ be the twist of this representation by the automorphism $\chi_{p,\bar{c}}$ of ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ defined via $e_{i,k} \mapsto c_i e_{i,k},\ f_{i,k} \mapsto c_i^{-1} f_{i,k},\ \psi_{i,k} \mapsto \psi_{i,k},\ q^{d_2} \mapsto q^{-\frac{p(n-p)}{2}} \cdot q^{d_2}$.*

Given a collection $\{(p_i, u_i, \bar{c}_i)\}_{i=1}^r$ with $0 \le p_i \le n-1, u_i \in \mathbb{C}^\times, \bar{c}_i \in (\mathbb{C}^\times)^{[n]}$, we say that it is *generic* if for any pair $1 \le i < i' \le r$, there are no $a, b \in \mathbb{Z}$ such that $b - a \equiv p_i - p_{i'} \pmod n$ and $u_i q_1^{b-a} q_2^{b+1} = u_{i'}$. We have the following simple result (see [FJMM1, Lemma 4.1]).

Lemma 1.7. *For a generic collection $\{(p_i, u_i, \bar{c}_i)\}_{i=1}^r$, the coproduct endows $\tau^{p_1}_{u_1,\bar{c}_1} \otimes \cdots \otimes \tau^{p_r}_{u_r,\bar{c}_r}$ with a structure of an ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-module. It is an irreducible lowest weight module.*

• *Macmahon modules* $M^{(p)}(u, K)$.

For $K \in \mathbb{C}^\times$, set $\phi^K(t) := \frac{K^{-1}t-K}{t-1}$. We say that K is *generic* if $K \notin q^{\mathbb{Z}}d^{\mathbb{Z}}$. For such K, the unique irreducible lowest weight ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-module of the lowest weight $(\phi^K(z/u)^{\delta_{i,p}})_{i\in[n]}$ is called the Macmahon module, denoted by $M^{(p)}(u, K)$. They were first studied in [FJMM1]. Recall that a collection of partitions $\bar{\lambda} = \{\lambda^{(j)}\}_{j\in\mathbb{N}}$ is called a *plane partition* if

$$\lambda_l^{(j)} \ge \lambda_l^{(j+1)} \text{ for all } j, l \in \mathbb{N} \text{ and } \lambda^{(j)} = \emptyset \text{ for } j \gg 0.$$

Proposition 1.8. [FJMM1, Theorem 4.3] *For a generic K, the vector space $M^{(p)}(u, K)$ has a basis $\{|\bar{\lambda}\rangle\}$ (labeled by all plane partitions) with $|\bar{\emptyset}\rangle$ being its lowest weight vector.*

In this paper, we will not need explicit formulas for the matrix coefficients in the basis $\{|\bar{\lambda}\rangle\}$.

1.6. **Vertex representations.**

In this section, we recall a family of vertex $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$-representations from [S], which generalize the classical Frenkel–Jing construction. Let S_n be the *generalized* Heisenberg algebra generated by $\{H_{i,k}|i\in[n],k\in\mathbb{Z}\backslash\{0\}\}$ and a central element H_0 with the defining relations

$$[H_{i,k},H_{j,l}]=d^{-km_{i,j}}\frac{[k]_q\cdot[ka_{i,j}]_q}{k}\delta_{k,-l}\cdot H_0.$$

Let S_n^+ be the subalgebra of S_n generated by $\{H_{i,k}|i\in[n],k>0\}\sqcup\{H_0\}$, and let $\mathbb{C}v_0$ be the S_n^+-representation with $H_{i,k}$ acting trivially and H_0 acting via the identity operator. The induced representation $F_n:=\mathrm{Ind}_{S_n^+}^{S_n}\mathbb{C}v_0$ is called the *Fock representation* of S_n.

We denote by $\{\bar{\alpha}_i\}_{i=1}^{n-1}$ the simple roots of $\mathfrak{sl}_n$, by $\{\bar{\Lambda}_i\}_{i=1}^{n-1}$ the fundamental weights of $\mathfrak{sl}_n$, by $\{\bar{h}_i\}_{i=1}^{n-1}$ the simple coroots of $\mathfrak{sl}_n$. Let $\bar{Q}:=\bigoplus_{i=1}^{n-1}\mathbb{Z}\bar{\alpha}_i$ be the root lattice of $\mathfrak{sl}_n$, $\bar{P}:=\bigoplus_{i=1}^{n-1}\mathbb{Z}\bar{\Lambda}_i=\bigoplus_{i=2}^{n-1}\mathbb{Z}\bar{\alpha}_i\oplus\mathbb{Z}\bar{\Lambda}_{n-1}$ be the weight lattice of $\mathfrak{sl}_n$. We also set

$$\bar{\alpha}_0:=-\sum_{i=1}^{n-1}\bar{\alpha}_i\in\bar{Q},\ \bar{\Lambda}_0:=0\in\bar{P},\ \bar{h}_0:=-\sum_{i=1}^{n-1}\bar{h}_i.$$

Let $\mathbb{C}\{\bar{P}\}$ be the $\mathbb{C}$-algebra generated by $e^{\bar{\alpha}_2},\ldots,e^{\bar{\alpha}_{n-1}},e^{\bar{\Lambda}_{n-1}}$ with the defining relations:

$$e^{\bar{\alpha}_i}\cdot e^{\bar{\alpha}_j}=(-1)^{\langle\bar{h}_i,\bar{\alpha}_j\rangle}e^{\bar{\alpha}_j}\cdot e^{\bar{\alpha}_i},\ e^{\bar{\alpha}_i}\cdot e^{\bar{\Lambda}_{n-1}}=(-1)^{\delta_{i,n-1}}e^{\bar{\Lambda}_{n-1}}\cdot e^{\bar{\alpha}_i}.$$

For $\alpha=\sum_{i=2}^{n-1}m_i\bar{\alpha}_i+m_n\bar{\Lambda}_{n-1}$, we define $e^{\bar{\alpha}}\in\mathbb{C}\{\bar{P}\}$ via

$$e^{\bar{\alpha}}:=(e^{\bar{\alpha}_2})^{m_2}\cdots(e^{\bar{\alpha}_{n-1}})^{m_{n-1}}(e^{\bar{\Lambda}_{n-1}})^{m_n}.$$

Let $\mathbb{C}\{\bar{Q}\}$ be the subalgebra of $\mathbb{C}\{\bar{P}\}$ generated by $\{e^{\bar{\alpha}_i}\}_{i=1}^{n-1}$.

For every $0\le p\le n-1$, define the space

$$W(p)_n:=F_n\otimes\mathbb{C}\{\bar{Q}\}e^{\bar{\Lambda}_p}.$$

Consider the operators $H_{i,l},e^{\bar{\alpha}},\partial_{\bar{\alpha}_i},z^{H_{i,0}},\mathrm{d}$ acting on $W(p)_n$, which assign to every element

$$v\otimes e^{\bar{\beta}}=(H_{i_1,-k_1}\cdots H_{i_N,-k_N}v_0)\otimes e^{\sum_{j=1}^{n-1}m_j\bar{\alpha}_j+\bar{\Lambda}_p}\in W(p)_n$$

the following values:

$$H_{i,l}(v\otimes e^{\bar{\beta}}):=(H_{i,l}v)\otimes e^{\bar{\beta}},\ e^{\bar{\alpha}}(v\otimes e^{\bar{\beta}}):=v\otimes e^{\bar{\alpha}}e^{\bar{\beta}},\ \partial_{\bar{\alpha}_i}(v\otimes e^{\bar{\beta}}):=\langle\bar{h}_i,\bar{\beta}\rangle v\otimes e^{\bar{\beta}},$$
$$z^{H_{i,0}}(v\otimes e^{\bar{\beta}}):=z^{\langle\bar{h}_i,\bar{\beta}\rangle}d^{\frac{1}{2}\sum_{j=1}^{n-1}\langle\bar{h}_i,m_j\bar{\alpha}_j\rangle m_{i,j}}v\otimes e^{\bar{\beta}},$$
$$\mathrm{d}(v\otimes e^{\bar{\beta}}):=(-\sum k_i+((\bar{\Lambda}_p,\bar{\Lambda}_p)-(\bar{\beta},\bar{\beta}))/2)v\otimes e^{\bar{\beta}}.$$

The following result provides a natural structure of an $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$-module on $W(p)_n$.

Proposition 1.9. [S, Proposition 3.2.2] *For any $\bar{c}=(c_0,\ldots,c_{n-1})\in(\mathbb{C}^\times)^{[n]}$, $u\in\mathbb{C}^\times$, and $0\le p\le n-1$, the following formulas define an action of $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ on $W(p)_n$:*

$$\rho^p_{u,\bar{c}}(e_i(z))=c_i\exp\left(\sum_{k>0}\frac{q^{-k/2}u^{-k}}{[k]_q}H_{i,-k}z^k\right)\exp\left(-\sum_{k>0}\frac{q^{-k/2}u^{k}}{[k]_q}H_{i,k}z^{-k}\right)e^{\bar{\alpha}_i}\left(\frac{z}{u}\right)^{1+H_{i,0}},$$

$$\rho^p_{u,\bar{c}}(f_i(z))=\frac{(-1)^{n\delta_{i,0}}}{c_i}\exp\left(-\sum_{k>0}\frac{q^{k/2}u^{-k}}{[k]_q}H_{i,-k}z^k\right)\exp\left(\sum_{k>0}\frac{q^{k/2}u^{k}}{[k]_q}H_{i,k}z^{-k}\right)e^{-\bar{\alpha}_i}\left(\frac{z}{u}\right)^{1-H_{i,0}},$$

$$\rho^p_{u,\bar{c}}(\psi_i^\pm(z))=\exp\left(\pm(q-q^{-1})\sum_{k>0}H_{i,\pm k}(z/u)^{\mp k}\right)\cdot q^{\pm\partial_{\bar{\alpha}_i}},\rho^p_{u,\bar{c}}(\gamma^{\pm1/2})=q^{\pm1/2},\rho^p_{u,\bar{c}}(q^{\pm d_1})=q^{\pm\mathrm{d}}$$

(note that the factor $(-1)^{n\delta_{i,0}}$ in $\rho^p_{u,\bar{c}}(f_i(z))$ appears due to the equality $(e^{\bar{\alpha}_i})^{-1}=(-1)^{n\delta_{i,0}}e^{-\bar{\alpha}_i}$).

1.7. **Shuffle algebra.**

Consider a $\mathbb{Z}_+^{[n]}$-graded $\mathbb{C}$-vector space

$$\mathbb{S} = \bigoplus_{\overline{k}=(k_0,\ldots,k_{n-1})\in\mathbb{Z}_+^{[n]}} \mathbb{S}_{\overline{k}},$$

where $\mathbb{S}_{(k_0,\ldots,k_{n-1})}$ consists of $\prod \mathfrak{S}_{k_i}$-symmetric rational functions in the variables $\{x_{i,j}\}_{i\in[n]}^{1\le j\le k_i}$. We also fix an $n\times n$ matrix of rational functions $\Omega = (\omega_{i,j}(z))_{i,j\in[n]} \in \mathrm{Mat}_{n\times n}(\mathbb{C}(z))$ by setting

$$\omega_{i,i}(z) = \frac{z-q^{-2}}{z-1},\ \omega_{i,i+1}(z) = \frac{d^{-1}z-q}{z-1},\ \omega_{i,i-1}(z) = \frac{z-qd^{-1}}{z-1},\ \text{and } \omega_{i,j}(z) = 1 \text{ otherwise.}$$

Let us now introduce the bilinear $\star$ product on $\mathbb{S}$: given $F\in\mathbb{S}_{\overline{k}}, G\in\mathbb{S}_{\overline{l}}$, define $F\star G\in\mathbb{S}_{\overline{k}+\overline{l}}$ by

$$(F\star G)(x_{0,1},\ldots,x_{0,k_0+l_0};\ldots;x_{n-1,1},\ldots,x_{n-1,k_{n-1}+l_{n-1}}) :=$$

$$\mathrm{Sym}_{\prod\mathfrak{S}_{k_i+l_i}}\left(F(\{x_{i,j}\}_{i\in[n]}^{1\le j\le k_i})G(\{x_{i',j'}\}_{i'\in[n]}^{k_{i'}<j'\le k_{i'}+l_{i'}})\cdot\prod_{i\in[n]}^{i'\in[n]}\prod_{j\le k_i}^{j'>k_{i'}}\omega_{i,i'}(x_{i,j}/x_{i',j'})\right).$$

This endows $\mathbb{S}$ with a structure of an associative unital algebra with the unit $\mathbf{1}\in\mathbb{S}_{(0,\ldots,0)}$. We will be interested only in a certain subspace of $\mathbb{S}$, defined by the *pole* and *wheel conditions*:

- We say that $F\in\mathbb{S}_{\overline{k}}$ satisfies the *pole conditions* if

$$F = \frac{f(x_{0,1},\ldots,x_{n-1,k_{n-1}})}{\prod_{i\in[n]}\prod_{j\le k_i}^{j'\le k_{i+1}}(x_{i,j}-x_{i+1,j'})},\ \text{where } f\in(\mathbb{C}[x_{i,j}^{\pm1}]_{i\in[n]}^{1\le j\le k_i})^{\prod\mathfrak{S}_{k_i}}.$$

- We say that $F\in\mathbb{S}_{\overline{k}}$ satisfies the *wheel conditions* if

$$F(\{x_{i,j}\}) = 0 \text{ once } x_{i,j_1}/x_{i+\epsilon,l} = qd^{\epsilon} \text{ and } x_{i+\epsilon,l}/x_{i,j_2} = qd^{-\epsilon} \text{ for some } \epsilon, i, j_1, j_2, l,$$

where $\epsilon\in\{\pm1\}, i\in[n], 1\le j_1,j_2\le k_i, 1\le l\le k_{i+\epsilon}$.

Let $S_{\overline{k}}\subset\mathbb{S}_{\overline{k}}$ be the subspace of all elements F satisfying the above two conditions and set

$$S := \bigoplus_{\overline{k}\in\mathbb{Z}_+^{[n]}} S_{\overline{k}}.$$

Further $S_{\overline{k}} = \oplus_{d\in\mathbb{Z}} S_{\overline{k},d}$ with $S_{\overline{k},d} := \{F\in S_{\overline{k}} | \mathrm{tot.deg}(F) = d\}$. The following is straightforward:

Lemma 1.10. *The subspace $S\subset\mathbb{S}$ is $\star$-closed.*

Now we are ready to introduce one of the key actors of this paper:

Definition 1.11. *The algebra $(S,\star)$ is called the shuffle algebra (of $A_{n-1}^{(1)}$-type).*

Recall the subalgebra $\ddot{U}^+$ of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ from Section 1.2. By standard results, $\ddot{U}^+$ is generated by $\{e_{i,k}\}_{i\in[n]}^{k\in\mathbb{Z}}$ with the defining relations (T2, T7.1). We equip the algebra $\ddot{U}^+$ with the $\mathbb{Z}^{[n]}\times\mathbb{Z}$–grading by assigning $\deg(e_{i,k}) = (1_i;k)$, where $1_i\in\mathbb{Z}^{[n]}$ is the vector with the ith coordinate 1 and all other coordinates being zero.

The following result is straightforward:

Proposition 1.12. *There exists a unique algebra homomorphism $\Psi:\ddot{U}^+\to\mathbb{S}$ such that $\Psi(e_{i,k}) = x_{i,1}^k\ \forall\ i\in[n], k\in\mathbb{Z}$.*

As a consequence, $\mathrm{Im}(\Psi)\subset S$. The following beautiful result was recently proved by Negut:

Theorem 1.13. [N, Theorem 1.1] *The homomorphism $\Psi:\ddot{U}^+\to S$ is an isomorphism of $\mathbb{Z}_+^{[n]}\times\mathbb{Z}$-graded algebras.*

1.8. **Shuffle bimodules.**

Following the ideas of [FJMM2], we introduce three families of S-bimodules.

• *Shuffle modules* $S_{1,p}(u)$.

For $u \in \mathbb{C}^\times$ and $0 \le p \le n-1$, consider a $\mathbb{Z}_+^{[n]}$-graded $\mathbb{C}$-vector space

$$S_{1,p}(u) = \bigoplus_{\overline{k}=(k_0,\ldots,k_{n-1})\in\mathbb{Z}_+^{[n]}} S_{1,p}(u)_{\overline{k}},$$

where the degree $\overline{k}$ component $S_{1,p}(u)_{\overline{k}}$ consists of all $\prod \mathfrak{S}_{k_i}$-symmetric rational functions F in the variables $\{x_{i,j}\}_{i\in[n]}^{1\le j\le k_i}$ satisfying the following three conditions:

(i) F satisfies the *pole conditions*, that is,

$$F = \frac{f(x_{0,1},\ldots,x_{n-1,k_{n-1}})}{\prod_{i\in[n]}\prod_{j\le k_i}^{j'\le k_{i+1}}(x_{i,j}-x_{i+1,j'})\cdot\prod_{j=1}^{k_p}(x_{p,j}-u)},\ \text{ where } f\in(\mathbb{C}[x_{i,j}^{\pm1}]_{i\in[n]}^{1\le j\le k_i})^{\prod\mathfrak{S}_{k_i}}.$$

(ii) F satisfies the *first kind wheel conditions*, that is,

$$F(\{x_{i,j}\})=0 \text{ once } x_{i,j_1}/x_{i+\epsilon,l}=qd^{\epsilon} \text{ and } x_{i+\epsilon,l}/x_{i,j_2}=qd^{-\epsilon} \text{ for some } \epsilon,i,j_1,j_2,l,$$

where $\epsilon\in\{\pm1\}, i\in[n], 1\le j_1,j_2\le k_i, 1\le l\le k_{i+\epsilon}$.

(iii) F satisfies the *second kind wheel conditions*, that is, it has a presentation as in (i) and

$$f(\{x_{i,j}\})=0 \text{ once } x_{p,j_1}=u \text{ and } x_{p,j_2}=q^2u \text{ for some } 1\le j_1,j_2\le k_p.$$

Fix $\overline{c}\in(\mathbb{C}^\times)^{[n]}$. Given $F\in S_{\overline{k}}$ and $G\in S_{1,p}(u)_{\overline{l}}$, we define $F\star G, G\star F\in S_{1,p}(u)_{\overline{k}+\overline{l}}$ via:

$$(1)\quad (F\star G)(x_{0,1},\ldots,x_{0,k_0+l_0};\ldots;x_{n-1,1},\ldots,x_{n-1,k_{n-1}+l_{n-1}}) := \prod_{i\in[n]} c_i^{k_i}\times$$

$$\mathrm{Sym}_{\prod\mathfrak{S}_{k_i+l_i}}\left(F(\{x_{i,j}\}_{i\in[n]}^{1\le j\le k_i})G(\{x_{i',j'}\}_{i'\in[n]}^{k_{i'}<j'\le k_{i'}+l_{i'}})\prod_{i\in[n]}^{i'\in[n]}\prod_{j\le k_i}^{j'>k_{i'}}\omega_{i,i'}(x_{i,j}/x_{i',j'})\prod_{j=1}^{k_p}\phi(x_{p,j}/u)\right)$$

and

$$(2)\quad (G\star F)(x_{0,1},\ldots,x_{0,k_0+l_0};\ldots;x_{n-1,1},\ldots,x_{n-1,k_{n-1}+l_{n-1}}) :=$$

$$\mathrm{Sym}_{\prod\mathfrak{S}_{k_i+l_i}}\left(G(\{x_{i,j}\}_{i\in[n]}^{1\le j\le l_i})F(\{x_{i',j'}\}_{i'\in[n]}^{l_{i'}<j'\le l_{i'}+k_{i'}})\prod_{i\in[n]}^{i'\in[n]}\prod_{j\le l_i}^{j'>l_{i'}}\omega_{i,i'}(x_{i,j}/x_{i',j'})\right).$$

These formulas endow $S_{1,p}(u)$ with a structure of an S-bimodule.

Identifying S with $'\ddot{U}^+\simeq\ddot{U}^+$ via Ψ, we get two commuting $'\ddot{U}^+$-actions on $S_{1,p}(u)$. Actually, we can extend one of them to an action of the whole algebra $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$:

Proposition 1.14. *The following formulas define an action of* $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ *on* $S_{1,p}(u)$*:*

$$\pi_{u,\overline{c}}^p(q^{d_2})G = q^{-\frac{p(n-p)}{2}+|\overline{k}|}\cdot G,\ \pi_{u,\overline{c}}^p(e_{i,k})G = x_i^k\star G,$$

$$\pi_{u,\overline{c}}^p(h_{i,0})G = (2k_i-k_{i-1}-k_{i+1}-\delta_{i,p})\cdot G,$$

$$\pi_{u,\overline{c}}^p(h_{i,l})G = \left(\frac{1}{l}\sum_{i'\in[n]}\sum_{j'=1}^{k_{i'}}[la_{i',i}]_q d^{lm_{i',i}}x_{i',j'}^l - \delta_{i,p}\frac{[l]_q}{l}q^lu^l\right)\cdot G \text{ for } l\ne0,$$

$$\pi_{u,\overline{c}}^p(f_{i,k})G = \frac{k_ic_i^{-1}}{q^{-1}-q}\left(\underset{z=0}{\mathrm{Res}}+\underset{z=\infty}{\mathrm{Res}}\right)\frac{z^kG(\{x_{i',j'}\}_{|x_{i,k_i}\mapsto z})}{\prod_{i'}\prod_{j'=1}^{k_{i'}-\delta_{i,i'}}\omega_{i',i}(\frac{x_{i',j'}}{z})}\frac{dz}{z}.$$

Here $k\in\mathbb{Z}, \bar{c}=(c_0,\ldots,c_{n-1})\in(\mathbb{C}^\times)^{[n]},\ G\in S_{1,p}(u)_{\overline{k}}$ *and* $|\overline{k}|:=\sum_{i\in[n]}k_i$.

Proof.

The only nontrivial relation is (T4) (compare to [FJMM2, Proposition 3.2]). However, we note that the k_i summands from the symmetrization appearing in $\pi^p_{u,\bar{c}}(e_{i,k})\pi^p_{u,\bar{c}}(f_{i,l})G$ cancel the k_i terms (out of k_i+1) from the symmetrization appearing in $\pi^p_{u,\bar{c}}(f_{i,l})\pi^p_{u,\bar{c}}(e_{i,k})G$. □

Remark 1.15. *Formulas for the action of* $h_{i,k}(k\in\mathbb{Z})$ *are equivalent to a single formula*

$$\pi^p_{u,\bar{c}}(\psi_i^\pm(z))G=\left(\prod_{j=1}^{k_i}\frac{q^2z-x_{i,j}}{z-q^2x_{i,j}}\cdot\prod_{j=1}^{k_{i+1}}\frac{z-qdx_{i+1,j}}{qz-dx_{i+1,j}}\cdot\prod_{j=1}^{k_{i-1}}\frac{dz-qx_{i-1,j}}{qdz-x_{i-1,j}}\cdot\phi(z/u)^{\delta_{i,p}}\right)^{\pm}\cdot G,$$

where $g(z)^\pm$ *denotes the expansion of a rational function* $g(z)$ *in* $z^{\mp1}$*, respectively.*

• *Shuffle modules* $S(\underline{u})$.

The above construction admits a "higher rank" generalization. For any $\overline{r}\in\mathbb{Z}_+^{[n]}$, consider

$$\underline{u}=(u_{0,1},\ldots,u_{0,r_0};\ldots;u_{n-1,1},\ldots,u_{n-1,r_{n-1}})\text{ with }u_{i,s}\in\mathbb{C}^\times.$$

Define $S(\underline{u})$ completely analogously to $S_{1,p}(u)$ with the following two modifications:
(i′) *Pole conditions* for a degree $\overline{k}$ function F should read as follows:

$$F=\frac{f(x_{0,1},\ldots,x_{n-1,k_{n-1}})}{\prod_{i\in[n]}\prod_{j\le k_i}^{j'\le k_{i+1}}(x_{i,j}-x_{i+1,j'})\cdot\prod_{i\in[n]}\prod_{s=1}^{r_i}\prod_{j=1}^{k_i}(x_{i,j}-u_{i,s})},\ f\in(\mathbb{C}[x_{i,j}^{\pm1}]_{i\in[n]}^{1\le j\le k_i})^{\prod\mathfrak{S}_{k_i}}.$$

(iii′) *Second kind wheel conditions* for such F should read as follows:

$$f(\{x_{i,j}\})=0\text{ once }x_{i,j_1}=u_{i,s}\text{ and }x_{i,j_2}=q^2u_{i,s}\text{ for some }i\in[n],1\le s\le r_i,1\le j_1,j_2\le k_i.$$

Let us endow $S(\underline{u})$ with an S-bimodule structure by applying formulas (1) and (2) with

$$\prod_{j=1}^{k_p}\phi(x_{p,j}/u)\rightsquigarrow\prod_{i\in[n]}\prod_{s=1}^{r_i}\prod_{j=1}^{k_i}\phi(x_{i,j}/u_{i,s}).$$

The resulting left ${}'\ddot{U}^+$-action on $S(\underline{u})$ can be extended to the ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action, denoted $\pi_{\underline{u},\bar{c}}$, given by the same formulas as in the case of $S_{1,p}(u)$ (see Proposition 1.14, Remark 1.15) except for the following changes in the action of $\psi_i^\pm(z)$ and q^{d_2}:

$$\phi(z/u)^{\delta_{i,p}}\rightsquigarrow\prod_{s=1}^{r_i}\phi(z/u_{i,s}),\ q^{-\frac{p(n-p)}{2}}\rightsquigarrow q^{-\sum_{p=0}^{n-1}r_p\cdot\frac{p(n-p)}{2}}.$$

Let $\mathbf{1}_{\underline{u}}$ denote the element $1\in S(\underline{u})_{(0,\ldots,0)}$. The following is obvious:

Lemma 1.16. *For* $X\in{}'\ddot{U}^+\cdot{}'\ddot{U}^0$*, we have* $\pi_{\underline{u},\bar{c}}(X)\mathbf{1}_{\underline{u}}=0$ *for all* $\underline{u},\bar{c}$ *if and only if* $X=0$.

• *Shuffle modules* $S^K_{1,p}(u)$ *and* $S^{\underline{K}}(\underline{u})$.

Another generalization of $S_{1,p}(u)$ is provided by the S-bimodules $S^K_{1,p}(u)$. As a vector space, $S^K_{1,p}(u)$ is defined similarly to $S_{1,p}(u)$ but without imposing the second kind wheel conditions. The S-bimodule structure on $S^K_{1,p}(u)$ is defined by the formulas (1) and (2) with the only change

$$\phi(t)\rightsquigarrow\phi^K(t):=(K^{-1}\cdot t-K)/(t-1).$$

The resulting left ${}'\ddot{U}^+$-action can be extended to the ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action on $S^K_{1,p}(u)$, denoted $\pi^{p,K}_{u,\bar{c}}$, given by the formulas from Proposition 1.14 and Remark 1.15 except for a change $\phi\rightsquigarrow\phi^K$.

It is clear how to define the "higher rank" generalization $S^{\underline{K}}(\underline{u})$, equip it with an S-bimodule structure, and extend the resulting left ${}'\ddot{U}^+$-action to the ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action $\pi^{\underline{K}}_{\underline{u},\bar{c}}$ on $S^{\underline{K}}(\underline{u})$.

2. Identification of representations

In this section, we establish relations between representations $\tau^p_{u,\bar{c}},\ \pi^p_{u,\bar{c}},\ \rho^p_{u,\bar{c}}$.

2.1. Isomorphism $\bar{S}_{1,p}(u)\simeq F^{(p)}(u)$.

Fix $0\leq p\leq n-1, u\in\mathbb{C}^\times, \bar{c}\in(\mathbb{C}^\times)^{[n]}$. Recall the action $\pi^p_{u,\bar{c}}$ of ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ on $S_{1,p}(u)$ from Proposition 1.14. Define

$$S':=\bigoplus_{\overline{k}\neq(0,\ldots,0)} S_{\overline{k}}\subset S.$$

Consider a $\mathbb{C}$-vector subspace

$$V_0:=S_{1,p}(u)\star S'=\mathrm{span}_{\mathbb{C}}\{G\star F|\ G\in S_{1,p}(u), F\in S'\}\subset S_{1,p}(u).$$

The following result is straightforward and its proof is left to the interested reader:

Lemma 2.1. *The subspace V_0 of $S_{1,p}(u)$ is invariant under the action $\pi^p_{u,\bar{c}}$ of ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$.*

Let $\bar{\pi}^p_{u,\bar{c}}$ denote the corresponding ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action on the factor space $\bar{S}_{1,p}(u):=S_{1,p}(u)/V_0$.

Theorem 2.2. *We have an isomorphism of ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-modules $\bar{\pi}^p_{u,\bar{c}}\simeq\tau^p_{u,\bar{c}}$.*

Proof.

By definition, $\tau^p_{u,\bar{c}}$ is an irreducible ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-representation generated by $|\emptyset\rangle$. Moreover, both $\bar{\mathbf{1}}_u\in\bar{S}_{1,p}(u)$ (the image of $\mathbf{1}_u$) and $|\emptyset\rangle\in F^{(p)}(u)$ are lowest weight vectors of the same lowest weight. Therefore, it suffices to estimate dimensions of the graded components of $\bar{S}_{1,p}(u)$:

$$(\heartsuit)\qquad \sum_{\overline{k}\in\mathbb{Z}_+^{[n]}}^{|\overline{k}|=m}\dim\bar{S}_{1,p}(u)_{\overline{k}}=p(m)\ \ \forall\ m\in\mathbb{N}\ \ (p(m):=\text{number of size } m \text{ partitions}).$$

Descending filtration.

To prove $(\heartsuit)$, we equip $S^m_{1,p}(u):=\bigoplus_{|\overline{k}|=m}S_{1,p}(u)_{\overline{k}}$ with a filtration $\{S^{m,\lambda}_{1,p}(u)\}_\lambda$ labeled by all size $\leq m$ partitions λ. We define $S^{m,\lambda}_{1,p}(u)$ via the specialization maps ρ_λ introduced below as

$$S^{m,\lambda}_{1,p}(u):=\bigcap_{\mu\succ\lambda}\mathrm{Ker}(\rho_\mu)\subset S^m_{1,p}(u),$$

where $\succ$ denotes the lexicographic order on the set of size $\leq m$ partitions.

Consider the $[n]$-coloring of the Young diagram λ by coloring the box $\square=(a,b)\in\lambda$ into $c(\square):=p-a+b\pmod n\in[n]$. Define

$$\overline{k}^\lambda:=(k_0^\lambda,\ldots,k_{n-1}^\lambda)\in\mathbb{Z}_+^{[n]},\ \text{where } k_i^\lambda=\#\{\square\in\lambda\mid c(\square)=i\}.$$

Remark 2.3. *We denote $\tau^p_{u,(1,\ldots,1)}$ simply by τ^p_u. Note that the map $|\lambda\rangle\mapsto\prod_{\square\in\lambda}c_{c(\square)}\cdot|\lambda\rangle$ induces an isomorphism of ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-representations $\tau^p_u\xrightarrow{\sim}\tau^p_{u,\bar{c}}$ for any $\bar{c}\in(\mathbb{C}^\times)^{[n]}$.*

Let us fill the boxes of λ by entering $q_1^aq_3^bu$ into the box $(a,b)\in\lambda$. For $F\in S_{1,p}(u)_{\overline{k}}$, we would like to specialize $\overline{k}^\lambda$ variables to the corresponding entries of λ. Such a naive substitution produces zeroes in numerators and denominators, so we need to modify it properly to get ρ_λ.

Specialization maps ρ_λ.

For $F\in S_{1,p}(u)_{\overline{k}}$, we set $\rho_\lambda(F)=0$ if $\overline{k}-\overline{k}^\lambda\notin\mathbb{Z}_+^{[n]}$. If $\bar{l}:=\overline{k}-\overline{k}^\lambda\in\mathbb{Z}_+^{[n]}$, we do the following:

∘ First, we consider the corner box $\square=(0,0)\in\lambda$ of color p and specialize $x_{p,k_p}\mapsto u$. Since F has the first order pole at $x_{p,k_p}=u$, the following is well-defined:

$$\rho^{(1)}_\lambda(F):=[(x_{p,k_p}-u)\cdot F]_{|x_{p,k_p}\mapsto u}.$$

$\circ$ Next, we specialize more variables to entries of the remaining boxes from the first row and the first column. For every box $\square=(a,0)\in\lambda$ $(0<a<\lambda_1)$ of color $p-a$, we choose the unspecified yet variable of the $(p-a)$-th family $\{x_{p-a,\cdot}\}$ and set it to $q_1^a u$. Likewise, for every box $\square=(0,b)\in\lambda$ $(0<b<\lambda'_1)$, we choose the unspecified yet variable of the $(p+b)$-th family $\{x_{p+b,\cdot}\}$ and set it to $q_3^b u$. We perform this procedure step-by-step moving from $(0,0)$ to the right and then from $(0,0)$ up. We denote the resulting specialization of F by $\rho_\lambda^{(\lambda_1+\lambda'_1-1)}(F)$.

$\circ$ If $(1,1)\notin\lambda$, we set $\rho_\lambda(F):=\rho_\lambda^{(\lambda_1+\lambda'_1-1)}(F)$. If λ contains $(1,1)$, we would like to specify another variable of the p-th family, say x_{p,k_p-1}, to q_1q_3u. Due to the first kind wheel conditions, the function $\rho_\lambda^{(\lambda_1+\lambda'_1-1)}(F)$ has zero at $x_{p,k_p-1}=q_1q_3u$. Hence, the following is well-defined:

$$\rho_\lambda^{(\lambda_1+\lambda'_1)}(F):=\left[\frac{1}{x_{p,k_p-1}-q_1q_3u}\cdot\rho_\lambda^{(\lambda_1+\lambda'_1-1)}(F)\right]_{|x_{p,k_p-1}\mapsto q_1q_3u}.$$

$\circ$ Next, we start moving from $(1,1)$ to the right and then from $(1,1)$ up. On each step, we specialize the corresponding $x_{\cdot,\cdot}$-variable to the prescribed entry of the diagram. However, due to the first kind wheel conditions, we have to eliminate order 1 zeros as above.

$\circ$ Performing this procedure $|\lambda|$ times, we finally obtain $\rho_\lambda^{(|\lambda|)}(F)\in\mathbb{C}(\{x_{i,j}\}_{i\in[n]}^{1\le j\le l_i})$. Set

$$\rho_\lambda(F):=\rho_\lambda^{(|\lambda|)}(F).$$

Key properties of ρ_λ.

Tracing back the contribution of the first and second kind wheel conditions, we find that

$$\rho_\lambda: S_{1,p}(u)_{\bar{k}^\lambda+\bar{l}}\longrightarrow S_{\bar{l}}\cdot G_{\bar{l},\lambda}:=\{F'\cdot G_{\bar{l},\lambda}|F'\in S_{\bar{l}}\},$$

$$G_{\bar{l},\lambda}=\frac{\prod_{j=1}^{l_p}(x_{p,j}-q^2u)\cdot\prod_{\square=(a,b)\in X_\lambda}\prod_{j=1}^{l_{c(\square)}}(x_{c(\square),j}-q_1^aq_3^bu)}{\prod_{j=1}^{l_p}(x_{p,j}-u)\cdot\prod_{\square=(a,b)\in\lambda}\left\{\prod_{j=1}^{l_{c(\square)-1}}(x_{c(\square)-1,j}-q_1^aq_3^bu)\prod_{j=1}^{l_{c(\square)+1}}(x_{c(\square)+1,j}-q_1^aq_3^bu)\right\}},$$

where the set $X_\lambda\subset\mathbb{Z}^2$ consists of those $(a,b)\in\mathbb{Z}^2$ such that $(a+1,b)\&(a+1,b+1)\in\lambda$, or $(a,b+1)\&(a+1,b+1)\in\lambda$, or $(a-1,b)\&(a-1,b-1)\in\lambda$, or $(a,b-1)\&(a-1,b-1)\in\lambda$.

For $F\in S_{1,p}^{|\lambda|+|\bar{l}|,\lambda}(u)_{\bar{k}^\lambda+\bar{l}}$, we further have $\rho_\lambda(F)\in S_{\bar{l}}\cdot G_{\bar{l},\lambda}Q_{\bar{l},\lambda}$, where

$$Q_{\bar{l},\lambda}=\prod_{j=1}^{l_{p-\lambda_1}}(x_{p-\lambda_1,j}-q_1^{\lambda_1}u)\cdot\prod_{b\ge1}^{\lambda_{b+1}<\lambda_b}\prod_{j=1}^{l_{p-\lambda_{b+1}+b}}(x_{p-\lambda_{b+1}+b,j}-q_1^{\lambda_{b+1}}q_3^bu).$$

The following two properties of ρ_λ are crucial:

Lemma 2.4. *(i) If $\bar{k}-\bar{k}^\lambda\notin\mathbb{Z}_+^{[n]}$, then $\rho_\lambda(S_{1,p}(u)_{\bar{k}}\star S_{\bar{l}})=0$ for any $\bar{l}\in\mathbb{Z}_+^{[n]}$.*
(ii) We have $\rho_\lambda(S_{1,p}^{|\bar{k}^\lambda+\bar{l}|,\lambda}(u)_{\bar{k}^\lambda+\bar{l}})=\rho_\lambda(S_{1,p}(u)_{\bar{k}^\lambda}\star S_{\bar{l}})$ for any $\bar{l}\in\mathbb{Z}_+^{[n]}$.

Proof of Lemma 2.4.

(i) For $F_1\in S_{1,p}(u)_{\bar{k}}$ and $F_2\in S_{\bar{l}}$, let us evaluate the ρ_λ-specialization of any summand from $F_1\star F_2$. In what follows, we say that $q_1^aq_3^bu$ gets into F_2 in the chosen summand if the $x_{\cdot,\cdot}$-variable which is specialized to $q_1^aq_3^bu$ enters F_2 rather than F_1. If u gets into F_2, we automatically get zero once we apply $\rho_\lambda^{(1)}$. A simple inductive argument shows that if at least one of the variables $\{q_1^au\}_{a=1}^{\lambda_1-1}\cup\{q_3^bu\}_{b=1}^{\lambda'_1-1}$ gets into F_2, we also obtain zero after applying $\rho_\lambda^{(a+1)}$ or $\rho_\lambda^{(\lambda_1+b)}$ since the corresponding $\omega_{\cdot,\cdot}$-factor is zero. If q_1q_3u gets into F_2, but all the entries from the first hook of λ get into F_1, then there are two zero $\omega_{\cdot,\cdot}$-factors, and so we get zero after applying $\rho_\lambda^{(\lambda_1+\lambda'_1)}$, etc. However, not all the specialized variables get into F_1 as $\bar{k}-\bar{k}^\lambda\notin\mathbb{Z}_+^{[n]}$. Hence, the ρ_λ-specialization of this summand is zero, and so $\rho_\lambda(F_1\star F_2)=0$.

(ii) For $F_1 \in S_{1,p}(u)_{\overline{k}^\lambda}, F_2 \in S_{\overline{l}}$, the specialization $\rho_\lambda(F_1 \star F_2)$ is a sum of ρ_λ-specializations applied to each summand from $F_1 \star F_2$. According to (i), only one such specialization is nonzero and we have $\rho_\lambda(F_1 \star F_2) = \rho_\lambda(F_1) \cdot F_2(\{x_{i,j}\}_{i\in[n]}^{1\leq j\leq l_i}) \cdot P$, where P denotes the product of the corresponding $\omega_{\cdot,\cdot}$-factors: $P = \prod_{\square=(a,b)\in\lambda} \prod_{i\in[n]} \prod_{j=1}^{l_i} \omega_{c(\square),i}(q_1^a q_3^b u/x_{i,j})$. It is straightforward to check that $P = \nu \cdot G_{\overline{l},\lambda} Q_{\overline{l},\lambda}$ with $\nu \in \mathbb{C}^\times$. To complete the proof of part (ii), it remains to provide $F_1 \in S_{1,p}(u)_{\overline{k}^\lambda}$ such that $\rho_\lambda(F_1) \neq 0$. To achieve this, we set $F_1 = K_{\bar{\lambda}'_{l(\lambda')}} \star \cdots \star K_{\bar{\lambda}'_1} \cdot \prod_{j=1}^{\overline{k}_p^\lambda} (x_{p,j} - u)^{-1}$ with $K_{\overline{r}} := \frac{\prod_{i\in[n]}\prod_{j\neq j'}(x_{i,j}-q^{-2}x_{i,j'})\cdot\prod_{i\in[n]}\prod_{j,j'}(x_{i,j}-q_1x_{i+1,j'})}{\prod_{i\in[n]}\prod_{j,j'}(x_{i,j}-x_{i+1,j'})}$ (the multidegree $\bar{\lambda}'_j \in \mathbb{Z}_+^{[n]}$ is prescribed by the coloring of the jth column of λ). □

Proof of (♡).

Now we are ready to deduce (♡), completing our proof of Theorem 2.2. Note that

$$\dim S_{1,p}^m(u) = \sum_{\lambda:|\lambda|\leq m} \dim \mathrm{gr}_\lambda(S_{1,p}^m(u)),\quad \dim \bar{S}_{1,p}^m(u) = \sum_{\lambda:|\lambda|\leq m} \dim \mathrm{gr}_\lambda(\bar{S}_{1,p}^m(u)),$$

where the filtration $\{\bar{S}_{1,p}^{m,\lambda}(u)\}_\lambda$ on $\bar{S}_{1,p}^m(u)$ is induced by the filtration $\{S_{1,p}^{m,\lambda}(u)\}_\lambda$ on $S_{1,p}^m(u)$. The ρ_λ-specialization identifies $\mathrm{gr}_\lambda(S_{1,p}^m(u))$ with $\rho_\lambda(S_{1,p}^{m,\lambda}(u))$. This observation and Lemma 2.4 imply that $\mathrm{gr}_\lambda(\bar{S}_{1,p}^m(u))$ is zero if $|\lambda| < m$ and is 1-dimensional if $|\lambda| = m$. This proves (♡). □

2.2. Generalizations to $S(\underline{u})$ and $S^{\underline{K}}(\underline{u})$.

The result of Theorem 2.2 can be generalized in both directions mentioned in Section 1.8. Recall the $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action $\pi_{\underline{u},\bar{c}}$ on the space $S(\underline{u})$, which preserves the subspace $S(\underline{u}) \star S'$ (compare to Lemma 2.1). Let $\bar{\pi}_{\underline{u},\bar{c}}$ be the induced $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action on $\bar{S}(\underline{u}) := S(\underline{u})/(S(\underline{u})\star S')$.

Theorem 2.5. *For generic $\{u_{i,s}\}$, we have an isomorphism of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-modules*

$$\bar{\pi}_{\underline{u},\bar{c}} \simeq \otimes_{i=0}^{n-1} \otimes_{s=1}^{r_i} \tau_{u_{i,s}}^i.$$

Proof.

The proof of this theorem goes along the same lines as for the case $\sum r_i = 1$ from above. For generic $u_{i,s}$, the tensor product $\otimes_{i=0}^{n-1} \otimes_{s=1}^{r_i} \tau_{u_{i,s}}^i$ is well-defined, irreducible, lowest weight representation generated by the lowest weight vector $|\emptyset\rangle_{\underline{u}} := \otimes_{i=0}^{n-1} \otimes_{s=1}^{r_i} |\emptyset\rangle$. On the other hand, the vector $\bar{\mathbf{1}}_{\underline{u}} \in \bar{S}(\underline{u})$ is the lowest weight vector of the same weight as $|\emptyset\rangle_{\underline{u}}$. Therefore, it suffices to compare the dimensions. This can be accomplished as above by using the specialization maps $\rho_{\underline{\lambda}}$ with $\underline{\lambda} = \{\lambda^{(0,1)}, \ldots, \lambda^{(n-1,r_{n-1})}\}$ (they are defined similarly to ρ_λ, but the entry of $\square = (a,b) \in \lambda^{(i,s)}$ is set to be $q_1^a q_3^b u_{i,s}$, while its color is $c(\square) := i - a + b \pmod n \in [n]$). □

Another generalization of Theorem 2.2 is provided by considering the representations $\pi_{\underline{u},\bar{c}}^{\underline{K}}$. Let $\bar{\pi}_{\underline{u},\bar{c}}^{\underline{K}}$ be the induced $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-action on the factor space $\bar{S}^{\underline{K}}(\underline{u}) := S^{\underline{K}}(\underline{u})/(S^{\underline{K}}(\underline{u}) \star S')$.

Theorem 2.6. *For generic $\{u_{i,s}, K_{i,s}\}$, we have an isomorphism of $'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-modules*

$$\bar{\pi}_{\underline{u},\bar{c}}^{\underline{K}} \simeq \otimes_{i=0}^{n-1} \otimes_{s=1}^{r_i} M^{(i)}(u_{i,s}, K_{i,s}).$$

Proof.

For simplicity, we explain the modifications required to carry out the case of $\bar{\pi}_{u,\bar{c}}^{p,K}$. The specialization maps $\rho_{\bar{\lambda}}$ are now parametrized by plane partitions $\bar{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots)$. We fill the boxes of $\bar{\lambda}$ by entering $q_1^a q_2^{j-1} q_3^b u$ into the box $\square = (a,b) \in \lambda^{(j)}$. The arguments from our proof of Theorem 2.2 apply word by word. Note that the only place where we used the second kind wheel conditions was the appearance of the factor $\prod_{j=1}^{l_p}(x_{p,j} - q^2 u)$ in $G_{\bar{l},\lambda}$. This is now compensated by a change of $Q_{\bar{l},\lambda}$–the factor which keeps track of the filtration depth. □

2.3. **Isomorphism $\rho^{p,\varpi}_{v,\bar{c}} \simeq \tau^{*,p}_u$.**

Given a representation ρ of an algebra B on a vector space V and an algebra homomorphism $\sigma: A \to B$, we use ρ^σ to denote the corresponding representation of A on V: $\rho^\sigma(x) = \rho(\sigma(x))$. To simplify our notation, we define ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-modules $\rho^{p,\varpi}_{v,\bar{c}} := (\rho^p_{v,\bar{c}})^\varpi$ and $\tau^{*,p}_u := {}^*(\tau^p_u)$. Actually, the left dual and the right dual modules of τ^p_u are isomorphic: ${}^*(\tau^p_u) \simeq (\tau^p_u)^*$.

Theorem 2.7. *For any $0 \leq p \leq n-1, v \in \mathbb{C}^\times, \bar{c} \in (\mathbb{C}^\times)^{[n]}$, we have an isomorphism of ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-modules $\rho^{p,\varpi}_{v,\bar{c}} \simeq \tau^{*,p}_u$, where $u := (-1)^{\frac{(n-2)(n-3)}{2}} q^{-1} d^{-p-(n-1)\delta_{p,0}} \cdot (c_0 \cdots c_{n-1})^{-1}$.*

Proof.

Our proof consists of three steps. First, we verify that vectors $v_0 \otimes e^{\bar{\Lambda}_p}$ and $|\emptyset\rangle^*$ have the same eigenvalues with respect to the "finite Cartan subalgebra" $\mathbb{C}[\psi^{\pm 1}_{0,0}, \ldots, \psi^{\pm 1}_{n-1,0}, q^{\pm d_2}]$. Second, we show that both of them are annihilated by $e_{i,k}$-action for any $i \in [n], k \in \mathbb{Z}$. Finally, we prove that both vectors have the same eigenvalues with respect to $\psi_{i,l}$-action ($\forall\ i \in [n], l \in \mathbb{Z}\backslash\{0\}$).

$\circ$ *Step 1.*

According to Proposition 1.9, elements $\psi_{i,0}, \gamma, q^{d_1}$ act on $v_0 \otimes e^{\bar{\Lambda}_p}$ via multiplication by $q^{\langle \bar{h}_i, \bar{\Lambda}_p\rangle}, q, 1$, respectively. Combining this with Proposition 1.4(a), we get

$$\rho^{p,\varpi}_{v,\bar{c}}(q^{d_2}) v_0 \otimes e^{\bar{\Lambda}_p} = q^{\frac{p(n-p)}{2}} \cdot v_0 \otimes e^{\bar{\Lambda}_p},\ \rho^{p,\varpi}_{v,\bar{c}}(\psi_{i,0}) v_0 \otimes e^{\bar{\Lambda}_p} = q^{\delta_{i,p}} \cdot v_0 \otimes e^{\bar{\Lambda}_p}\quad \forall\ i \in [n].$$

We also have $\tau^p_u(q^{d_2})|\emptyset\rangle = q^{-\frac{p(n-p)}{2}} \cdot |\emptyset\rangle$ and $\tau^p_u(\psi_{i,0})|\emptyset\rangle = q^{-\delta_{i,p}} \cdot |\emptyset\rangle$. Therefore, the vectors $v_0 \otimes e^{\bar{\Lambda}_p} \in \rho^{p,\varpi}_{v,\bar{c}}$ and $|\emptyset\rangle^* \in \tau^{*,p}_u$ have the same weight with respect to $\mathbb{C}[\psi^{\pm 1}_{0,0}, \ldots, \psi^{\pm 1}_{n-1,0}, q^{\pm d_2}]$.

Remark 2.8. *This explains the appearance of $q^{-\frac{p(n-p)}{2}}$ in the formulas for $\tau^p_{u,\bar{c}}(q^{d_2}),\ \pi^p_{u,\bar{c}}(q^{d_2})$.*

$\circ$ *Step 2.*

First, we prove $\rho^{p,\varpi}_{v,\bar{c}}(e_{i,0}) v_0 \otimes e^{\bar{\Lambda}_p} = 0$ for any $i \in [n]$. For $i \neq 0$, this is clear as $\langle \bar{h}_i, \bar{\Lambda}_p\rangle + 1 > 0$, while $H_{i',k} v_0 = 0$ for all $i' \in [n], k > 0$. For $i = 0$, by Proposition 1.4(a)

$$\varpi(e_{0,0}) = d\gamma\psi_{0,0}[\cdots[f_{1,1}, f_{2,0}]_q, \cdots, f_{n-1,0}]_q.$$

We claim that

$$\rho^p_{v,\bar{c}}([\cdots[f_{1,1}, f_{2,0}]_q, \cdots, f_{n-1,0}]_q) v_0 \otimes e^{\bar{\Lambda}_p} = 0.$$

Writing down this multicommutator explicitly, we see that each summand of it has a form $f_{i_{n-1},j_{n-1}} \cdots f_{i_1,j_1}$ with $\{i_1, \ldots, i_{n-1}\} = [n]^\times$ and $j_k = \delta_{i_k,1}$. Since $-\langle \bar{h}_i, \bar{\Lambda}_p\rangle + 1 > 0$ for $i \neq p$, we see that $\rho^p_{v,\bar{c}}(f_{i_1,j_1}) v_0 \otimes e^{\bar{\Lambda}_p} = 0$ unless $i_1 = p \neq 1$. For $i_1 = p \neq 1$, we get $\rho^p_{v,\bar{c}}(f_{i_1,j_1}) v_0 \otimes e^{\bar{\Lambda}_p} = \pm c^{-1}_{i_1} v_0 \otimes e^{\bar{\Lambda}^{(1)}_p}$ with $\bar{\Lambda}^{(1)}_p := \bar{\Lambda}_p - \bar{\alpha}_{i_1}$. The key property of this weight is $-\langle \bar{h}_i, \bar{\Lambda}^{(1)}_p\rangle + 1 \geq 0$. In particular, we see that $\rho^p_{v,\bar{c}}(f_{i_2,j_2}) v_0 \otimes e^{\bar{\Lambda}^{(1)}_p} = 0$ unless $i_2 = p-1 \neq 1$ or $i_2 = p+1 \neq 1$. In the latter two cases, the result is $\pm c^{-1}_{i_2} v_0 \otimes e^{\bar{\Lambda}^{(2)}_p}$ with $\bar{\Lambda}^{(2)}_p := \bar{\Lambda}^{(1)}_p - \bar{\alpha}_{i_2}$ satisfying a similar property. Continuing in the same way, we finally get to the kth place with $i_k = 1$ and $j_k = 1$. As $-\langle \bar{h}_1, \bar{\Lambda}^{(k-1)}_p\rangle + 1 \geq 0$, we have

$$\rho^p_{v,\bar{c}}(f_{i_k,j_k} \cdots f_{i_1,j_1}) v_0 \otimes e^{\bar{\Lambda}_p} = 0.$$

This completes our proof of the equality $\rho^{p,\varpi}_{v,\bar{c}}(e_{i,0}) v_0 \otimes e^{\bar{\Lambda}_p} = 0$ for all $i \in [n]$.

According to (‡) from the next step, we have $\rho^{p,\varpi}_{v,\bar{c}}(h_{j,\pm 1}) v_0 \otimes e^{\bar{\Lambda}_p} = 0$ for $j \neq p$. Combining this formula with the relation (T5′) $[h_{j,\pm 1}, e_{i,k}] = d^{\mp m_{j,i}} \gamma^{-1/2} [\pm a_{j,i}]_q \cdot e_{i,k\pm 1}$, one gets

$$\rho^{p,\varpi}_{v,\bar{c}}(e_{i,k}) v_0 \otimes e^{\bar{\Lambda}_p} = 0\ \text{ for any }\ i \in [n], k \in \mathbb{Z}.$$

On the other hand, the identity $S(e_i(z)) = -\psi^-_i(\gamma^{-1/2}z)^{-1} e_i(\gamma^{-1}z)$ combined with the formulas from Proposition 1.5 imply a similar equality $\tau^{*,p}_u(e_{i,k})|\emptyset\rangle^* = 0$ for any $i \in [n], k \in \mathbb{Z}$.

∘ *Step 3.*

Let us now prove that both $v_0\otimes e^{\bar{\Lambda}_p}\in \rho^{p,\varpi}_{v,\bar{c}}$ and $|\emptyset\rangle^*\in \tau^{*,p}_u$ are eigenvectors with respect to the generators $\psi_{i,l}$ $(l\neq 0)$ and have the same eigenvalues. By the definition of τ^p_u, we have

$\tau^{*,p}_u(\psi^\pm_i(z))|\emptyset\rangle^*=\phi(z/u)^{-\delta_{i,p}}|\emptyset\rangle^*\Rightarrow \tau^{*,p}_u(\psi_{i,\pm r})|\emptyset\rangle^*=\pm\delta_{i,p}(q-q^{-1})(q^2u)^{\pm r}|\emptyset\rangle^*\ \forall\, i\in[n], r\in\mathbb{N}.$

Therefore, it remains to show

$$(\ddagger)\qquad \rho^{p,\varpi}_{v,\bar{c}}(\psi_{i,\pm r})v_0\otimes e^{\bar{\Lambda}_p}=\pm\delta_{i,p}(q-q^{-1})(q^2u)^{\pm r}v_0\otimes e^{\bar{\Lambda}_p}\ \text{for all}\ i\in[n], r\in\mathbb{N}.$$

Our proof of ($\ddagger$) is based on the technical lemma:

Lemma 2.9. *We have the following equalities:*

$$\rho^{p,\varpi}_{v,\bar{c}}(f_{i,0})v_0\otimes e^{\bar{\Lambda}_p}=\delta_{i,p}c_p^{-1}\cdot\lambda^{\delta_{p,0}}\cdot v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p},\tag{3}$$

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{i,-1})v_0\otimes e^{\bar{\Lambda}_p}=\delta_{i,p}q^{-1}u^{-1}\cdot v_0\otimes e^{\bar{\Lambda}_p},\tag{4}$$

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{i,1})v_0\otimes e^{\bar{\Lambda}_p}=\delta_{i,p}qu\cdot v_0\otimes e^{\bar{\Lambda}_p},\tag{5}$$

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{i,-1})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}=-\delta_{i,p}q^{-3}u^{-1}\cdot v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p},\tag{6}$$

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{i,1})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}=-\delta_{i,p}q^{3}u\cdot v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p},\tag{7}$$

where $\mathfrak{c}:=\prod_{j\in[n]}c_j,\ \lambda:=(-1)^{\frac{(n-2)(n-3)}{2}}v^{-1}q^{-1}d^{-1}\mathfrak{c},\ u:=(-1)^{\frac{(n-2)(n-3)}{2}}q^{-1}d^{-p-(n-1)\delta_{p,0}}\mathfrak{c}^{-1}.$

Proof of Lemma 2.9.

∘ For $i\neq 0$, we have $\varpi(f_{i,0})=f_{i,0}$ and $-\langle\bar{h}_i,\bar{\Lambda}_p\rangle+1=1-\delta_{i,p}\geq 0$, so that

$$\rho^{p,\varpi}_{v,\bar{c}}(f_{i,0})v_0\otimes e^{\bar{\Lambda}_p}=\delta_{i,p}c_p^{-1}\cdot v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}.\ ✓$$

Recalling the formula for $\varpi(f_{0,0})$ from Proposition 1.4(a), we get

$$\rho^{p,\varpi}_{v,\bar{c}}(f_{0,0})v_0\otimes e^{\bar{\Lambda}_p}=q^{-\delta_{p,0}}d^{-1}\cdot\rho^p_{v,\bar{c}}([e_{n-1,0},\cdots,[e_{2,0},e_{1,-1}]_{q^{-1}}\cdots]_{q^{-1}})v_0\otimes e^{\bar{\Lambda}_p}.$$

As $\rho^p_{v,\bar{c}}(e_{j,0})v_0\otimes e^{\bar{\Lambda}_p}=0$ for $j\neq 0$, we see (by rewriting the above multicommutator) that

$$\rho^p_{v,\bar{c}}([e_{n-1,0},\cdots,[e_{2,0},e_{1,-1}]_{q^{-1}}\cdots]_{q^{-1}})v_0\otimes e^{\bar{\Lambda}_p}=\rho^p_{v,\bar{c}}(e_{n-1,0})\cdots\rho^p_{v,\bar{c}}(e_{2,0})\rho^p_{v,\bar{c}}(e_{1,-1})v_0\otimes e^{\bar{\Lambda}_p}.$$

For $p\neq 0$, the same argument as before implies $\rho^p_{v,\bar{c}}(e_{p,0})\cdots\rho^p_{v,\bar{c}}(e_{2,0})\rho^p_{v,\bar{c}}(e_{1,-1})v_0\otimes e^{\bar{\Lambda}_p}=0$, while for $p=0$ we have

$$\rho^p_{v,\bar{c}}(e_{n-1,0})\cdots\rho^p_{v,\bar{c}}(e_{2,0})\rho^p_{v,\bar{c}}(e_{1,-1})v_0\otimes e^{\bar{\Lambda}_p}=v^{-1}(c_1\cdots c_{n-1})\cdot v_0\otimes(e^{\bar{\alpha}_{n-1}}\cdots e^{\bar{\alpha}_1}).$$

Since $e^{\bar{\alpha}_{n-1}}\cdots e^{\bar{\alpha}_1}=(-1)^{\frac{(n-2)(n-3)}{2}}e^{-\bar{\alpha}_0}$, we finally get

$$\rho^{0,\varpi}_{v,\bar{c}}(f_{0,0})v_0\otimes e^0=(-1)^{\frac{(n-2)(n-3)}{2}}v^{-1}q^{-1}d^{-1}\mathfrak{c}\cdot c_0^{-1}v_0\otimes e^{-\bar{\alpha}_0}.\ ✓$$

In what follows below, we assume $p\neq 0$.

∘ Combining the formula for $\varpi(h_{0,-1})$ from Proposition 1.4(c) with

$$\rho^p_{v,\bar{c}}(e_{0,1})v_0\otimes e^{\bar{\Lambda}_p}=\rho^p_{v,\bar{c}}(e_{2,0})v_0\otimes e^{\bar{\Lambda}_p}=\cdots=\rho^p_{v,\bar{c}}(e_{n-1,0})v_0\otimes e^{\bar{\Lambda}_p}=0,$$

we get

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{0,-1})v_0\otimes e^{\bar{\Lambda}_p}=(-1)^n d^{n-1}\cdot\rho^p_{v,\bar{c}}(e_{0,1})\rho^p_{v,\bar{c}}(e_{n-1,0})\cdots\rho^p_{v,\bar{c}}(e_{2,0})\rho^p_{v,\bar{c}}(e_{1,-1})v_0\otimes e^{\bar{\Lambda}_p}.$$

The latter is zero, since $\rho^p_{v,\bar{c}}(e_{p-1,0})\cdots\rho^p_{v,\bar{c}}(e_{1,-1})v_0\otimes e^{\bar{\Lambda}_p}=\pm v^{-1}c_1\cdots c_{p-1}\cdot v_0\otimes e^{\bar{\Lambda}_p+\bar{\alpha}_1+\cdots+\bar{\alpha}_{p-1}}$ and $\langle\bar{h}_p,\bar{\Lambda}_p+\bar{\alpha}_1+\cdots+\bar{\alpha}_{p-1}\rangle+1>0$. Thus $\rho^{p,\varpi}_{v,\bar{c}}(h_{0,-1})v_0\otimes e^{\bar{\Lambda}_p}=0$ for $p\neq 0$. ✓

For $i\neq 0$, the formula for $\varpi(h_{i,-1})$ combined with $\rho^p_{v,\bar{c}}(e_{j,0})v_0\otimes e^{\bar{\Lambda}_p}=0$ $(j\neq 0)$ implies

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{i,-1})v_0\otimes e^{\bar{\Lambda}_p}=(-1)^{i+1}d^i\rho^p_{v,\bar{c}}(e_{i,0})\cdots\rho^p_{v,\bar{c}}(e_{1,0})\rho^p_{v,\bar{c}}(e_{i+1,0})\cdots\rho^p_{v,\bar{c}}(e_{n-1,0})\rho^p_{v,\bar{c}}(e_{0,0})v_0\otimes e^{\bar{\Lambda}_p}.$$

If $i<p$, then $\langle\bar{h}_p,\bar{\Lambda}_p+\bar{\alpha}_0+\sum_{j=p+1}^{n-1}\bar{\alpha}_j\rangle+1>0$ and so

$$\rho^p_{v,\bar{c}}(e_{p,0})\cdots\rho^p_{v,\bar{c}}(e_{n-1,0})\rho^p_{v,\bar{c}}(e_{0,0})v_0\otimes e^{\bar{\Lambda}_p}=0\Rightarrow\rho^{p,\varpi}_{v,\bar{c}}(h_{i,-1})v_0\otimes e^{\bar{\Lambda}_p}=0.$$

If $i>p$, then $\langle\bar{h}_p,\bar{\Lambda}_p+\bar{\alpha}_0+\sum_{j=i+1}^{n-1}\bar{\alpha}_j+\sum_{j=1}^{p-1}\bar{\alpha}_j\rangle+1>0$ and hence $\rho^{p,\varpi}_{v,\bar{c}}(h_{i,-1})v_0\otimes e^{\bar{\Lambda}_p}=0$. If $i=p$, then we get

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{p,-1})v_0\otimes e^{\bar{\Lambda}_p}=(-1)^{p+1}d^p(c_0\cdots c_{n-1})v_0\otimes(e^{\bar{\alpha}_p}\cdots e^{\bar{\alpha}_1}e^{\bar{\alpha}_{p+1}}\cdots e^{\bar{\alpha}_{n-1}}e^{\bar{\alpha}_0}e^{\bar{\Lambda}_p})=$$
$$(-1)^{\frac{(n-2)(n-3)}{2}}d^p\mathfrak{c}\cdot v_0\otimes e^{\bar{\Lambda}_p}\quad\text{as}\quad e^{\bar{\alpha}_p}\cdots e^{\bar{\alpha}_1}e^{\bar{\alpha}_{p+1}}\cdots e^{\bar{\alpha}_{n-1}}e^{\bar{\alpha}_0}=(-1)^{\frac{n(n-1)}{2}+p}\cdot e^0.\ \checkmark$$

◦ By the same argument, we have $\rho^{p,\varpi}_{v,\bar{c}}(h_{i,1})v_0\otimes e^{\bar{\Lambda}_p}=0$ if $i\neq p$. For $i=p$, we get

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{p,1})v_0\otimes e^{\bar{\Lambda}_p}=(-1)^{n+p+1}d^{-p}(c_0^{-1}\cdots c_{n-1}^{-1})v_0\otimes(e^{-\bar{\alpha}_0}e^{-\bar{\alpha}_{n-1}}\cdots e^{-\bar{\alpha}_{p+1}}e^{-\bar{\alpha}_1}\cdots e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p})=$$
$$(-1)^{\frac{(n-2)(n-3)}{2}}d^{-p}\mathfrak{c}^{-1}\cdot v_0\otimes e^{\bar{\Lambda}_p}\quad\text{as}\quad e^{-\bar{\alpha}_0}e^{-\bar{\alpha}_{n-1}}\cdots e^{-\bar{\alpha}_{p+1}}e^{-\bar{\alpha}_1}\cdots e^{-\bar{\alpha}_p}=(-1)^{\frac{n(n+1)}{2}+p}\cdot e^0.\ \checkmark$$

◦ By the same argument, we have $\rho^{p,\varpi}_{v,\bar{c}}(h_{i,-1})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}=0$ if $i\neq p$. For $i=p$, only one summand in the multicommutator acts nontrivially:

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{p,-1})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}=$$
$$(-1)^{p+1}d^p(-q^{-2})\rho^p_{v,\bar{c}}(e_{p-1,0})\cdots\rho^p_{v,\bar{c}}(e_{1,0})\rho^p_{v,\bar{c}}(e_{p+1,0})\cdots\rho^p_{v,\bar{c}}(e_{0,0})\rho^p_{v,\bar{c}}(e_{p,0})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}=$$
$$(-1)^{1+\frac{(n-2)(n-3)}{2}}d^pq^{-2}\mathfrak{c}\cdot v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}.\ \checkmark$$

◦ By the same argument, we have $\rho^{p,\varpi}_{v,\bar{c}}(h_{i,1})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}=0$ if $i\neq p$. For $i=p$, only one summand in the multicommutator acts nontrivially:

$$\rho^{p,\varpi}_{v,\bar{c}}(h_{p,1})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}=$$
$$(-1)^{n+p+1}d^{-p}(-q^2)\rho^p_{v,\bar{c}}(f_{p,0})\rho^p_{v,\bar{c}}(f_{0,0})\cdots\rho^p_{v,\bar{c}}(f_{p+1,0})\rho^p_{v,\bar{c}}(f_{1,0})\cdots\rho^p_{v,\bar{c}}(f_{p-1,0})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}=$$
$$(-1)^{1+\frac{(n-2)(n-3)}{2}}d^{-p}q^2\mathfrak{c}^{-1}\cdot v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}.\ \checkmark$$

The proofs of (4)–(7) for $p=0$ are analogous and are left to the interested reader. □

Note that $\rho^{p,\varpi}_{v,\bar{c}}(e_{p,0})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}=c_p\lambda^{-\delta_{p,0}}\cdot v_0\otimes e^{\bar{\Lambda}_p}$. Combining this with the identity $[h_{p,\pm1},e_{p,\pm r}]=\gamma^{-1/2}(q+q^{-1})e_{p,\pm(r+1)}$ and the equalities (4)–(7) from Lemma 2.9, we get

$$\rho^{p,\varpi}_{v,\bar{c}}(e_{p,\pm r})v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}=c_p\lambda^{-\delta_{p,0}}(q^2u)^{\pm r}\cdot v_0\otimes e^{\bar{\Lambda}_p}\ \text{for}\ r\in\mathbb{N}.$$

On the other hand, we have

$$\rho^{p,\varpi}_{v,\bar{c}}(\psi_{i,\pm r})=\pm(q-q^{-1})[\rho^{p,\varpi}_{v,\bar{c}}(e_{i,\pm r}),\rho^{p,\varpi}_{v,\bar{c}}(f_{i,0})]\ \text{for}\ r\in\mathbb{N}.$$

Since $\rho^{p,\varpi}_{v,\bar{c}}(e_{i,\pm r})v_0\otimes e^{\bar{\Lambda}_p}=\rho^{p,\varpi}_{v,\bar{c}}(f_{i,0})v_0\otimes e^{\bar{\Lambda}_p}=0$ for $i\neq p$, we get $\rho^{p,\varpi}_{v,\bar{c}}(\psi_{i,\pm r})v_0\otimes e^{\bar{\Lambda}_p}=0$ if $i\neq p$. The equality (‡) follows now from

$$\rho^{p,\varpi}_{v,\bar{c}}(\psi_{p,\pm r})v_0\otimes e^{\bar{\Lambda}_p}=\pm(q-q^{-1})\rho^{p,\varpi}_{v,\bar{c}}(e_{p,\pm r})\rho^{p,\varpi}_{v,\bar{c}}(f_{p,0})v_0\otimes e^{\bar{\Lambda}_p}=\pm(q-q^{-1})(q^2u)^{\pm r}v_0\otimes e^{\bar{\Lambda}_p}.$$

The irreducibility of $\rho^p_{v,\bar{c}}$ and τ^p_u implies that both ${}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$-representations $\rho^{p,\varpi}_{v,\bar{c}}$ and $\tau^{*,p}_u$ are irreducible. These representations are generated by the vectors $v_0\otimes e^{\bar{\Lambda}_p}$ and $|\emptyset\rangle^*$, which are the highest weight vectors of the same weight, due to Steps 1–3. Theorem 2.7 follows. □

Corollary 2.10. *For any* $0\leq p\leq n-1,\ v,v'\in\mathbb{C}^\times,\ \bar{c},\bar{c}'\in(\mathbb{C}^\times)^{[n]}$ *with* $\prod_{i\in[n]}c_i=\prod_{i\in[n]}c'_i$, *we have an isomorphism of* $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$*-representations* $\rho^p_{v,\bar{c}}\simeq\rho^p_{v',\bar{c}'}$.

3. Matrix elements of L operators

In this section, we study the matrix elements of L operators associated to $\rho^p_{u,\bar{c}}$. Let us denote $\rho^p_{1,\bar{c}}$ simply by $\rho^p_{\bar{c}}$. As $\rho^p_{u,\bar{c}} \xrightarrow{\sim} \rho^p_{\bar{c}}\ \forall\ u \in \mathbb{C}^\times$ by Corollary 2.10, it suffices to work only with $\rho^p_{\bar{c}}$. We provide a new realization of the S-bimodule $S_{1,p}(u)$ as a bimodule generated by $L^{p,\bar{c}}_{\emptyset,\emptyset}$.

3.1. **Matrix elements.**

For any $w \in W(p)^*_n$ and $v \in W(p)_n$, we consider

$$L^{p,\bar{c}}_{w,v} := \langle 1 \otimes w | (1 \otimes \rho^p_{\bar{c}})(R') | 1 \otimes v \rangle,$$

the matrix element of the universal R-matrix R' with respect to the second component. We will mainly work with the cases $v = |\emptyset\rangle := v_0 \otimes e^{\bar{\Lambda}_p} \in W(p)_n$ or $w = \langle\emptyset|$–the dual of $|\emptyset\rangle$. In what follows, we abbreviate $|\emptyset\rangle$ and $\langle\emptyset|$ simply by $\emptyset$ when they appear as indexes of matrix elements.

Lemma 3.1. *For any $i \in [n], r \in \mathbb{N}, v \in W(p)_n$, we have*

$$[h_{i,-r}, L^{p,\bar{c}}_{\emptyset,v}]_{q^{-r}} = (\gamma/q)^{r/2} \cdot L^{p,\bar{c}}_{\emptyset,\rho^p_{\bar{c}}(h_{i,-r})v}.$$

Proof.

It suffices to combine $\Delta(h_{i,-r}) = h_{i,-r} \otimes \gamma^{-r/2} + \gamma^{r/2} \otimes h_{i,-r}$ with $R'\Delta(h_{i,-r}) = \Delta^{\mathrm{op}}(h_{i,-r})R'$, and compare the matrix elements between $1 \otimes \langle\emptyset|$ and $1 \otimes v$ in the latter equality. □

Our first goal is to compute explicitly $L^{p,\bar{c}}_{\emptyset,\emptyset}$. The shuffle-type formula for $L^{p,\bar{c}}_{\emptyset,\emptyset}$ was obtained in [FT1, Theorem 4.8(a)]. To state the result, let $\Psi^{\geq} : \ddot{U}'^{\geq} \xrightarrow{\sim} S^{\geq}$ be the natural extension of the isomorphism $\Psi : \ddot{U}'^{+} \xrightarrow{\sim} S$ from Theorem 1.13.

Theorem 3.2. *The image of $L^{p,\bar{c}}_{\emptyset,\emptyset}$ under $\Psi^{\geq}$ has the following form:*

$$\Psi^{\geq}(L^{p,\bar{c}}_{\emptyset,\emptyset}) = \sum_{N=0}^{\infty} a_{p,N} \mathfrak{c}^{-N} q^{-d_1} q^{\bar{\Lambda}_p} \Gamma^0_{p,N},$$

where $a_{p,0} = 1, a_{p,N} \in \mathbb{C}[q^{\pm 1}, d^{\pm 1}]$ and the shuffle elements $\Gamma^0_{p,N} \in S_{(N,\ldots,N)}$ are defined via

$$\Gamma^0_{p,N} = \frac{\prod_{i\in[n]} \prod_{1\leq j\neq j'\leq N} (x_{i,j} - q^{-2} x_{i,j'}) \cdot \prod_{i\in[n]} \prod_{j=1}^{N} x_{i,j}}{\prod_{i\in[n]} \prod_{1\leq j,j'\leq N} (x_{i,j} - x_{i+1,j'})} \cdot \prod_{j=1}^{N} \frac{x_{0,j}}{x_{p,j}}.$$

Recall the Hopf pairing $'\varphi : {}'\ddot{U}^{\geq} \times {}'\ddot{U}^{\leq} \to \mathbb{C}$ from Theorem 1.1(d). Clearly, the generators $h_{j,r}$ ($r \in \mathbb{N}$) are orthogonal to all the generators of ${}'\ddot{U}^{\geq}$ except for $h_{i,-r}$. Moreover, we have

$$'\varphi(h_{i,-r}, h_{j,r}) = \frac{[ra_{i,j}]_q d^{rm_{i,j}}}{r(q - q^{-1})}.$$

Definition 3.3. *Let $\{h^{\perp}_{i,r}\}_{i\in[n]}$ be the basis of* $\mathrm{span}_{\mathbb{C}}\langle h_{0,-r}, \cdots, h_{n-1,-r}\rangle$*, which is dual to $\{h_{i,r}\}_{i\in[n]}$ with respect to the above pairing. In other words, $'\varphi(h^{\perp}_{i,r}, h_{j,s}) = \delta_{i,j}\delta_{r,s}$.*

The first insight towards the elements $\Psi^{-1}(\Gamma^0_{p,N})$ is given by the following result:

Lemma 3.4. *We have $\Psi^{-1}(\Gamma^0_{p,1}) = -(q^{-1} - q)^{-n} \varpi(h^{\perp}_{p,1})$.*

Proof.

Applying Ψ to the formulas for $\varpi(h_{k,-1})$ from Proposition 1.4(b,c), we find:

$$\Psi(\varpi(h_{k,-1})) = (q^{-1} - q)^{n-1} \cdot \frac{\prod_{i\in[n]} x_i}{\prod_{i\in[n]}(x_i - x_{i+1})} \cdot \left\{ (q + q^{-1}) \frac{x_0}{x_k} - d^{-1} \frac{x_0}{x_{k+1}} - d \frac{x_0}{x_{k-1}} \right\}.$$

Rewriting this as $\Psi(\varpi(h_{k,-1})) = -(q^{-1} - q)^n \sum_{p\in[n]} {}'\varphi(h_{k,-1}, h_{p,1}) \Gamma^0_{p,1}$, we get the claim. □

Now we are ready to state the main result of this section.

Theorem 3.5. *Given $\bar{c}\in(\mathbb{C}^\times)^{[n]}$, define $u\in\mathbb{C}^\times$ as in Theorem 2.7 via*

$$u:=(-1)^{\frac{(n-2)(n-3)}{2}}q^{-1}d^{-p-(n-1)\delta_{p,0}}\mathfrak{c}^{-1} \text{ with } \mathfrak{c}=c_0\cdots c_{n-1}.$$

(i) For any $i\neq p$, we have

$$\varpi(e_i(z))\cdot L^{p,\bar{c}}_{\emptyset,\emptyset}=L^{p,\bar{c}}_{\emptyset,\emptyset}\cdot\varpi(e_i(z)),$$
$$\varpi(f_i(z))\cdot L^{p,\bar{c}}_{\emptyset,\emptyset}=L^{p,\bar{c}}_{\emptyset,\emptyset}\cdot\varpi(f_i(z)),$$
$$\varpi(\psi^\pm_i(z))\cdot L^{p,\bar{c}}_{\emptyset,\emptyset}=L^{p,\bar{c}}_{\emptyset,\emptyset}\cdot\varpi(\psi^\pm_i(z)).$$

(ii) We have

$$(z-u)\cdot\varpi(e_p(z))\cdot L^{p,\bar{c}}_{\emptyset,\emptyset}=L^{p,\bar{c}}_{\emptyset,\emptyset}\cdot\varpi(e_p(z))\cdot(q^{-1}z-qu),$$
$$(q^{-1}z-qu)\cdot\varpi(f_p(z))\cdot L^{p,\bar{c}}_{\emptyset,\emptyset}=L^{p,\bar{c}}_{\emptyset,\emptyset}\cdot\varpi(f_p(z))\cdot(z-u),$$
$$\varpi(\psi^\pm_p(z))\cdot L^{p,\bar{c}}_{\emptyset,\emptyset}=L^{p,\bar{c}}_{\emptyset,\emptyset}\cdot\varpi(\psi^\pm_p(z)).$$

(iii) We have the following explicit formula

$$(\sharp)\qquad L^{p,\bar{c}}_{\emptyset,\emptyset}=q^{-d_1}q^{\bar{\Lambda}_p}\exp\left(\sum_{r=1}^{\infty}\frac{[r]_q}{r}(qu)^r\varpi(h^\perp_{p,r})\right).$$

Remark 3.6. *An analogous computation for the representation $\tau^{*,p}_u$ is much simpler. The corresponding matrix element $L^{\tau^{*,p}_u}_{\emptyset,\emptyset}:=\langle 1\otimes\emptyset|(1\otimes\tau^{*,p}_u)({}'R)|1\otimes\emptyset\rangle$ equals*

$$\langle 1\otimes\emptyset|(1\otimes\tau^{*,p}_u)(q^{\frac{1}{n}(d_2-\sum_{j=1}^{n-1}\bar{\Lambda}_j)\otimes c'+c'\otimes\frac{1}{n}(d_2-\sum_{j=1}^{n-1}\bar{\Lambda}_j)+\sum_{j=1}^{n-1}\bar{\Lambda}_j\otimes h_{j,0}}\cdot\exp(\sum_{i\in[n]}\sum_{r=1}^{\infty}h^\perp_{i,r}\otimes h_{i,r}))|1\otimes\emptyset\rangle,$$

since $\tau^p_u(f_{i,k})|\emptyset\rangle=0\ \forall\ i,k$. As $\tau^{*,p}_u(h_{i,r})|\emptyset\rangle^*=\delta_{i,p}\frac{[r]_q}{r}q^ru^r|\emptyset\rangle^*$, $\tau^{*,p}_u(h_{i,0})|\emptyset\rangle^*=\delta_{i,p}|\emptyset\rangle^*$, we get

$$L^{\tau^{*,p}_u}_{\emptyset,\emptyset}=q^{\frac{1}{n}(d_2-\sum_{j=1}^{n-1}\bar{\Lambda}_j)+\bar{\Lambda}_p}\exp\left(\sum_{r=1}^{\infty}\frac{[r]_q}{r}q^ru^rh^\perp_{p,r}\right).$$

Thus, $\varpi(L^{\tau^{*,p}_u}_{\emptyset,\emptyset})$ coincides with the RHS of ($\sharp$). However, we are not aware of the conceptual reason for $L^{p,\bar{c}}_{\emptyset,\emptyset}=\varpi(L^{\tau^{*,p}_u}_{\emptyset,\emptyset})$ (though it would significantly simplify our proof of Theorem 3.5).

3.2. **Proof of Theorem 3.5.**

Our proof is based on the equality

$$(\star)\qquad \langle 1\otimes w|R'\Delta(x)|1\otimes v\rangle=\langle 1\otimes w|\Delta^{\mathrm{op}}(x)R'|1\otimes v\rangle$$

for any $x\in\ddot{U}'_{q,d}(\mathfrak{sl}_n),\ v\in W(p)_n,\ w\in W(p)^*_n$.

Notation: Given a collection of elements $\beta_1,\cdots,\beta_N\in\{\pm\bar{\alpha}_0,\cdots,\pm\bar{\alpha}_{n-1}\}$ and $0\leq p\leq n-1$, consider $v_0\otimes e^{\beta_1}\cdots e^{\beta_N}e^{\bar{\Lambda}_p}$–an element of $W(p)_n$. We will also use the same notation for a dual element from $W(p)^*_n$, when writing it in the matrix coefficients of L operators.

• *Case $p\neq 0$.*

(i) We need to show that $L^{p,\bar{c}}_{\emptyset,\emptyset}$ commutes with $\varpi(e_{i,k}),\varpi(f_{i,k}),\varpi(h_{i,k})$ for any $k\in\mathbb{Z},i\neq p$.

◦ *Proof of $[L^{p,\bar{c}}_{\emptyset,\emptyset},\varpi(e_{i,0})]=0$ and $[L^{p,\bar{c}}_{\emptyset,\emptyset},\varpi(f_{i,0})]=0$ for $i\neq 0,p$.*

By construction, we have

$$\Delta(e_{i,k})=e_{i,k}\otimes 1+\psi^{-1}_{i,0}\gamma^{-k}\otimes e_{i,k}+\sum_{r\in\mathbb{N}}\psi_{i,-r}\gamma^{-k-r/2}\otimes e_{i,k+r},$$

$$\Delta(f_{i,k}) = 1\otimes f_{i,k} + f_{i,k}\otimes \psi_{i,0}\gamma^{-k} + \sum_{r\in\mathbb{N}} f_{i,k-r}\otimes \psi_{i,r}\gamma^{-k+r/2}.$$

Evaluating both sides of $(\star)$ at $v=|\emptyset\rangle, w=\langle\emptyset|$ and $x=e_{i,0}$ or $x=f_{i,0}$, we immediately get $[L^{p,\bar{c}}_{\emptyset,\emptyset}, e_{i,0}]=0$ and $[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{i,0}]=0$. It remains to use $\varpi(e_{i,0})=e_{i,0},\ \varpi(f_{i,0})=f_{i,0}$ for $i\neq 0$. ✓

∘ *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,-1})]=0$ *and* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,1})]=0$.

Evaluating both sides of $(\star)$ at $v=|\emptyset\rangle, w=\langle\emptyset|$ and $x=e_{0,1}$ or $x=f_{0,-1}$, we immediately get $[L^{p,\bar{c}}_{\emptyset,\emptyset}, e_{0,1}]=0$ and $[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{0,-1}]=0$, respectively. It remains to apply the equalities $\varpi(e_{0,-1})=(-d)^n e_{0,1}$ and $\varpi(f_{0,1})=(-d)^{-n}f_{0,-1}$ from Proposition 1.4(d). ✓

∘ *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(h_{i,-1})]=0$ *for any* $i\in[n]$.

It suffices to prove $[\Psi^{\geq}(L^{p,\bar{c}}_{\emptyset,\emptyset}), \Psi^{\geq}(\varpi(h_{i,-1}))]=0$. According to Lemma 3.4, $\Psi(\varpi(h_{i,-1}))$ is a linear combination of $\Gamma^0_{p',1}$. On the other hand, $\Psi^{\geq}(L^{p,\bar{c}}_{\emptyset,\emptyset})$ is a linear combination of $q^{-d_1}q^{\bar{\Lambda}_p}\Gamma^0_{p,N}$. The commutativity of the elements $\{\Gamma^0_{p',N}\}^{N\in\mathbb{N}}_{p'\in[n]}$ has been established in [FT1], while $q^{-d_1}q^{\bar{\Lambda}_p}$ obviously commutes with $\Gamma^0_{p',1}$. The result follows. ✓

∘ *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(h_{i,1})]=0$ *for any* $i\neq 0,p$.

According to Proposition 1.4(b), it suffices to prove that $C=0$, where C is defined via

(8) $$C := [L^{p,\bar{c}}_{\emptyset,\emptyset}; [f_{i,0},[f_{i-1,0},\cdots,[f_{1,0},[f_{i+1,0},\cdots,[f_{n-1,0},f_{0,0}]_{q^{-1}}\cdots]_{q^{-1}}]_{q^{-1}}\cdots]_{q^{-1}}]_{q^{-2}}]_1.$$

In what follows, we assume $i<p\leq n-1$ leaving the case $0<p<i$ to the interested reader. Applying iteratively the commutator identity

($\diamondsuit$) $$[a,[b,c]_u]_v = [[a,b]_x,c]_{uv/x} + x\cdot[b,[a,c]_{v/x}]_{u/x}$$

together with $[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{j,0}]=0$ for $j\neq 0,p$, we reduce to a stronger equality $C^{(1)}_1+C^{(1)}_2=0$ with

$$C^{(1)}_1 := [[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{p,0}]_{q^{-1}}, [f_{p+1,0},\cdots,[f_{n-1,0},f_{0,0}]_{q^{-1}}\cdots]_{q^{-1}}]_1,$$

$$C^{(1)}_2 := q^{-1}[f_{p,0},[f_{p+1,0},\cdots,[f_{n-1,0},[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{0,0}]_q]_{q^{-1}}\cdots]_{q^{-1}}]_1.$$

Evaluating both sides of $(\star)$ for appropriate v,w,x step-by-step, we obtain an explicit formula

$$C^{(1)}_2 = -(-q)^{p-n}\cdot\frac{\psi_{p+1,0}\cdots\psi_{n-1,0}\psi_{0,0}}{c_p\cdots c_{n-1}c_0}\cdot L^{p,\bar{c}}_{v_0\otimes e^{\bar{\alpha}_{p+1}}\cdots e^{\bar{\alpha}_0}e^{\bar{\Lambda}_p}, v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}}.$$

Let us now compute $C^{(1)}_1$. Evaluating both sides of $(\star)$ at $v=|\emptyset\rangle,\ w=\langle\emptyset|,\ x=f_{p,0}$, we find

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{p,0}]_{q^{-1}} = -q^{-1}c_p^{-1}\cdot L^{p,\bar{c}}_{\emptyset, v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}}.$$

Evaluating both sides of $(\star)$ at $v=v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}, w=\langle\emptyset|, x=f_{j,0}$, we find $[L^{p,\bar{c}}_{\emptyset,v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}}, f_{j,0}]=0$ for $p+1<j\leq n-1$. Applying iteratively ($\diamondsuit$), we get $C^{(1)}_1=C^{(2)}_1+C^{(2)}_2$ with

$$C^{(2)}_1 := -q^{-1}c_p^{-1}[[L^{p,\bar{c}}_{\emptyset,v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}}, f_{p+1,0}]_{q^{-1}}, [f_{p+2,0},\cdots,[f_{n-1,0},f_{0,0}]_{q^{-1}}\cdots]_{q^{-1}}]_1,$$

$$C^{(2)}_2 := -q^{-2}c_p^{-1}[f_{p+1,0},[f_{p+2,0},\cdots,[f_{n-1,0},[L^{p,\bar{c}}_{\emptyset,v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}}, f_{0,0}]_q]_{q^{-1}}\cdots]_{q^{-1}}]_1.$$

Evaluating both sides of $(\star)$ for appropriate v,w,x step-by-step, we obtain an explicit formula

$$C^{(2)}_2 = (-q)^{p-n}\cdot\frac{\psi_{p+1,0}\cdots\psi_{n-1,0}\psi_{0,0}}{c_p\cdots c_{n-1}c_0}\cdot L^{p,\bar{c}}_{v_0\otimes e^{\bar{\alpha}_{p+1}}\cdots e^{\bar{\alpha}_0}e^{\bar{\Lambda}_p}, v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}}$$
$$-(-q)^{p-n}\cdot\frac{\psi_{p+2,0}\cdots\psi_{n-1,0}\psi_{0,0}}{c_p\cdots c_{n-1}c_0}\cdot L^{p,\bar{c}}_{v_0\otimes e^{\bar{\alpha}_{p+2}}\cdots e^{\bar{\alpha}_0}e^{\bar{\Lambda}_p}, v_0\otimes e^{-\bar{\alpha}_{p+1}}e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}}.$$

The first summand cancels $C_2^{(1)}$, while the second summand is very similar to $C_2^{(1)}$.

Evaluating $C_1^{(2)}$, we get a similar formula $C_1^{(2)} = C_1^{(3)} + C_2^{(3)}$ with

$$C_2^{(3)} = (-q)^{p-n} \cdot \frac{\psi_{p+2,0}\cdots\psi_{n-1,0}\psi_{0,0}}{c_p\cdots c_{n-1}c_0} \cdot L^{p,\bar{c}}_{v_0\otimes e^{\bar{\alpha}_{p+2}}\cdots e^{\bar{\alpha}_0}e^{\bar{\Lambda}_p},\, v_0\otimes e^{-\bar{\alpha}_{p+1}}e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}}$$
$$- (-q)^{p-n} \cdot \frac{\psi_{p+3,0}\cdots\psi_{n-1,0}\psi_{0,0}}{c_p\cdots c_{n-1}c_0} \cdot L^{p,\bar{c}}_{v_0\otimes e^{\bar{\alpha}_{p+3}}\cdots e^{\bar{\alpha}_0}e^{\bar{\Lambda}_p},\, v_0\otimes e^{-\bar{\alpha}_{p+2}}e^{-\bar{\alpha}_{p+1}}e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p}}.$$

Proceeding further in the same way, we see that all nontrivial summands in the formula for C split into pairs of opposite terms. Hence, $C = 0$ and so $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(h_{i,1})] = 0$ for $i \neq 0, p$. ✓

∘ *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{i,k})] = 0$ *and* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{i,k})] = 0$ *for any* $k \in \mathbb{Z}, i \neq p$.

Choose $j \neq 0, p$ such that $a_{j,i} \neq 0$. Combining the commutator relations

$$[\varpi(h_{j,\pm1}), \varpi(e_{i,k})] = d^{\mp m_{j,i}}[a_{j,i}]_q \cdot \varpi(e_{i,k\pm1}),\ [\varpi(h_{j,\pm1}), \varpi(f_{i,k})] = -d^{\mp m_{j,i}}[a_{j,i}]_q \cdot \varpi(f_{i,k\pm1})$$

with

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{i,-\delta_{i,0}})] = 0,\ [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{i,\delta_{i,0}})] = 0,\ [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(h_{j,\pm1})] = 0,$$

we get the equalities $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{i,k})] = 0$ and $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{i,k})] = 0$ by induction on k. ✓

∘ *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(\psi_{i,k})] = 0$ *for any* $k \in \mathbb{Z}, i \neq p$.

For $k \neq 0$, this follows immediately from the defining relation (T4) and the previous step. For $k = 0$, it suffices to prove $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \psi_{i,0}] = 0$ for any $i \in [n]$. This equality follows by evaluating both sides of ($\star$) at $w = \langle\emptyset|,\ v = |\emptyset\rangle,\ x = \psi_{i,0}$. ✓

(ii) The first two equalities from part (ii) can be rewritten as follows:

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,k+1})]_q = q^2u \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,k})]_{q^{-1}} \quad \forall\ k \in \mathbb{Z}, \tag{9}$$

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,k-1})]_q = u^{-1} \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,k})]_{q^{-1}} \quad \forall\ k \in \mathbb{Z}. \tag{10}$$

It suffices to check (9) and (10) for single values of k as we can derive the general equalities by commuting further iteratively with $\varpi(h_{p+1,\pm1})$.

∘ *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,1})]_q = q^2u \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,0})]_{q^{-1}}$.

Evaluating both sides of ($\star$) at $v = |\emptyset\rangle,\ w = \langle\emptyset|,\ x = e_{p,0}$, we find $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,0})]_{q^{-1}} = c_p \cdot L^{p,\bar{c}}_{v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p},\emptyset}$. As $\varpi(e_{p,0}) = e_{p,0}$, we finally get

$$q^2u \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,0})]_{q^{-1}} = q^2c_pu \cdot L^{p,\bar{c}}_{v_0\otimes e^{-\bar{\alpha}_p}e^{\bar{\Lambda}_p},\emptyset}.$$

To compute $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,1})]_q$, let us first evaluate $\varpi(e_{p,1})$. Due to (T5′), we have

$$[h_{p,1}, e_{p,0}] = (q+q^{-1})e_{p,1} \Rightarrow \varpi(e_{p,1}) = -(q+q^{-1})^{-1} \cdot [\varpi(e_{p,0}), \varpi(h_{p,1})],$$

where $\varpi(e_{p,0}) = e_{p,0}$ and

$$\varpi(h_{p,1}) = (-1)^{n+p}d^{-p}q^n \cdot [f_{p,0}, \cdots, [f_{1,0}, [f_{p+1,0}, \cdots, [f_{n-1,0}, f_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-2}}.$$

Applying iteratively the equality (♢) together with the relation (T4), we finally get

$$\varpi(e_{p,1}) = (-1)^{n+p+1}d^{-p}q^{n-2}\psi_{p,0}[f_{p-1,0}, \cdots, [f_{1,0}, [f_{p+1,0}, \cdots, [f_{n-1,0}, f_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}}$$

Therefore, it remains to evaluate

$$C := [L^{p,\bar{c}}_{\emptyset,\emptyset}, [f_{p-1,0}, \cdots, [f_{1,0}, [f_{p+1,0}, \cdots, [f_{n-1,0}, f_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}}]_q.$$

Applying iteratively the equality (♢) together with $[L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{j,0}] = 0$ for $j \neq 0, p$, we get

$$C = [f_{p-1,0}, \cdots, [f_{1,0}, [f_{p+1,0}, \cdots, [f_{n-1,0}, [L^{p,\bar{c}}_{\emptyset,\emptyset}, f_{0,0}]_q]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}}.$$

To compute this multicommutator, we apply the equality $(\star)$ with an appropriate choice of v, w, x step-by-step. Leaving details to the interested reader, let is present the final formula

$$C = (-1)^n q^{3-n} \prod_{i \neq p} \frac{\psi_{i,0}}{c_i} \cdot L^{p,\bar{c}}_{v_0 \otimes e^{\bar{\alpha}_{p-1}} \cdots e^{\bar{\alpha}_1} e^{\bar{\alpha}_{p+1}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} e^{\bar{\Lambda}_p}, \emptyset}.$$

Since $\prod_{i \in [n]} \psi_{i,0} = 1$ in $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$, we finally get

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,1})]_q = (-1)^{\frac{(n-2)(n-3)}{2}} d^{-p} q c_p \mathfrak{c}^{-1} \cdot L^{p,\bar{c}}_{v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}, \emptyset},$$

where we used the following identity in $\mathbb{C}\{\bar{P}\}$

$$e^{\bar{\alpha}_{p-1}} \cdots e^{\bar{\alpha}_1} e^{\bar{\alpha}_{p+1}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} = (-1)^{\frac{n(n-1)}{2} + p} e^{-\bar{\alpha}_p}.$$

The equality $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,1})]_q = q^2 u \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{p,0})]_{q^{-1}}$ follows. ✓

∘ *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,-1})]_q = u^{-1} \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,0})]_{q^{-1}}$.

Evaluating both sides of $(\star)$ at $v = |\emptyset\rangle,\ w = \langle \emptyset|,\ x = f_{p,0}$, we find $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,0})]_{q^{-1}} = \frac{-1}{qc_p} \cdot L_{\emptyset, v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}$. As $\varpi(f_{p,0}) = f_{p,0}$, we finally get

$$u^{-1} \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,0})]_{q^{-1}} = -q^{-1} c_p^{-1} u^{-1} \cdot L^{p,\bar{c}}_{\emptyset, v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}.$$

To evaluate $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,-1})]_q$, let us first compute $\varpi(f_{p,-1})$. Due to (T5′), we have

$$[h_{p,-1}, f_{p,0}] = -(q+q^{-1}) f_{p,-1} \Rightarrow \varpi(f_{p,-1}) = (q+q^{-1})^{-1} \cdot [\varpi(f_{p,0}), \varpi(h_{p,-1})],$$

where $\varpi(f_{p,0}) = f_{p,0}$ and

$$\varpi(h_{p,-1}) = (-1)^{p+1} d^p \cdot [e_{p,0}, \cdots, [e_{1,0}, [e_{p+1,0}, \cdots, [e_{n-1,0}, e_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-2}}.$$

Applying iteratively the equality (♢) together with the relation (T4), we finally get

$$\varpi(f_{p,-1}) = (-1)^{p+1} d^p \cdot [e_{p-1,0}, \cdots, [e_{1,0}, [e_{p+1,0}, \cdots, [e_{n-1,0}, e_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}} \cdot \psi_{p,0}^{-1}.$$

Therefore, it remains to evaluate

$$C := [L^{p,\bar{c}}_{\emptyset,\emptyset}, [e_{p-1,0}, \cdots, [e_{1,0}, [e_{p+1,0}, \cdots, [e_{n-1,0}, e_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}}]_q.$$

Applying iteratively the equality (♢) together with $[L^{p,\bar{c}}_{\emptyset,\emptyset}, e_{j,0}] = 0$ for $j \neq 0, p$, we get

$$C = [e_{p-1,0}, \cdots, [e_{1,0}, [e_{p+1,0}, \cdots, [e_{n-1,0}, [L^{p,\bar{c}}_{\emptyset,\emptyset}, e_{0,0}]_q]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}}.$$

To compute this multicommutator, we apply the equality $(\star)$ with an appropriate choice of v, w, x step-by-step. Leaving details to the interested reader, let is present the final formula

$$C = -c_p^{-1} \mathfrak{c} \cdot L^{p,\bar{c}}_{\emptyset, v_0 \otimes e^{\bar{\alpha}_{p-1}} \cdots e^{\bar{\alpha}_1} e^{\bar{\alpha}_{p+1}} \cdots e^{\bar{\alpha}_{n-1}} e^{\bar{\alpha}_0} e^{\bar{\Lambda}_p}} \cdot \psi_{p,0}.$$

Therefore,

$$[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,-1})]_q = (-1)^{\frac{n(n-1)}{2}} d^p c_p^{-1} \mathfrak{c} \cdot L^{p,\bar{c}}_{\emptyset, v_0 \otimes e^{-\bar{\alpha}_p} e^{\bar{\Lambda}_p}}.$$

The equality $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,-1})]_q = u^{-1} \cdot [L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{p,0})]_{q^{-1}}$ follows. ✓

∘ *Proof of* $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(\psi_p^{\pm}(z))] = 0$.

Define $\widetilde{\psi}_{p,N} \in {}'\ddot{U}_{q,d}(\mathfrak{sl}_n)$ as the coefficient of z^{-N} in $\psi_p^+(z) - \psi_p^-(z)$, so that $[e_{p,a}, f_{p,b}] = \frac{\widetilde{\psi}_{p,a+b}}{q-q^{-1}}$ for any $a, b \in \mathbb{Z}$. Set $X_N := [\varpi(\widetilde{\psi}_{p,N}), L^{p,\bar{c}}_{\emptyset,\emptyset}]$. Combining the equalities

$$(\varpi(e_{p,k+1}) - u\varpi(e_{p,k}))L^{p,\bar{c}}_{\emptyset,\emptyset} = L^{p,\bar{c}}_{\emptyset,\emptyset}(q^{-1}\varpi(e_{p,k+1}) - qu\varpi(e_{p,k})),$$

$$(q^{-1}\varpi(f_{p,l+1}) - qu\varpi(f_{p,l}))L^{p,\bar{c}}_{\emptyset,\emptyset} = L^{p,\bar{c}}_{\emptyset,\emptyset}(\varpi(f_{p,l+1}) - u\varpi(f_{p,l})),$$

we get the following recursive relation $q^{-1}X_{k+l+2} - u(q+q^{-1})X_{k+l+1} + u^2qX_{k+l} = 0$.

As $X_{-1} = X_0 = 0$, we get $X_k = 0$ for any $k \in \mathbb{Z}$. This proves $[L^{p,\bar{c}}_{\emptyset,\emptyset}, \varpi(\psi_p^{\pm}(z))] = 0$. ✓

(iii) The unique element satisfying conditions (i, ii) of Theorem 3.5 and whose shuffle interpretation has a form as in Theorem 3.2 (we only need to know that it lives in an appropriate completion and its 'purely Cartan part' equals $q^{-d_1}q^{\bar{\Lambda}_p}$) is given by the formula (♯).

- *Case* $p = 0$.

Parts (i) and (iii) are proved completely analogously to the case $p \neq 0$. Since the last equality in (ii) follows from the former two, it suffices to check (9) and (10) for some $k \in \mathbb{Z}$.

∘ *Proof of* $[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,0})]_q = q^2u \cdot [L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,-1})]_{q^{-1}}$.

According to Proposition 1.4(d), we have $\varpi(e_{0,-1}) = (-d)^n e_{0,1}$. Evaluating both sides of $(\star)$ at $v = |\emptyset\rangle,\ w = \langle\emptyset|,\ x = e_{0,1}$, we get $[L^{0,\bar{c}}_{\emptyset,\emptyset}, e_{0,1}]_{q^{-1}} = (-1)^n c_0 \cdot L^{0,\bar{c}}_{v_0\otimes e^{-\bar{\alpha}_0},\emptyset}$. Therefore

$$q^2u \cdot [L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,-1})]_{q^{-1}} = uq^2d^nc_0 \cdot L^{0,\bar{c}}_{v_0\otimes e^{-\bar{\alpha}_0},\emptyset}.$$

Next, we evaluate the left-hand side of the claimed equality. According to Proposition 1.4(a)

$$\varpi(e_{0,0}) = d(-q)^{n-2}\gamma\psi_{0,0} \cdot [f_{n-1,0}, \cdots, [f_{2,0}, f_{1,1}]_{q^{-1}} \cdots]_{q^{-1}}.$$

Applying iteratively the equality $(\diamondsuit)$ together with $[L^{0,\bar{c}}_{\emptyset,\emptyset}, f_{j,0}] = 0$ for $j \neq 0$, we get

$$[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,0})]_q = d(-q)^{n-2}\gamma\psi_{0,0} \cdot [f_{n-1,0}, \cdots, [f_{2,0}, [L^{0,\bar{c}}_{\emptyset,\emptyset}, f_{1,1}]_q]_{q^{-1}} \cdots]_{q^{-1}}.$$

Evaluating this multicommutator step-by-step as before, we finally get

$$[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,0})]_q = qdc_0\mathfrak{c}^{-1} \cdot L^{0,\bar{c}}_{v_0\otimes e^{\bar{\alpha}_{n-1}}\cdots e^{\bar{\alpha}_1},\emptyset} = (-1)^{\frac{(n-2)(n-3)}{2}} qdc_0\mathfrak{c}^{-1} \cdot L^{0,\bar{c}}_{v_0\otimes e^{-\bar{\alpha}_0},\emptyset}.$$

The equality $[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,0})]_q = q^2u \cdot [L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(e_{0,-1})]_{q^{-1}}$ follows. ✓

∘ *Proof of* $[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,0})]_q = u^{-1}[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,1})]_{q^{-1}}$.

According to Proposition 1.4(d), we have $\varpi(f_{0,1}) = (-d)^{-n} f_{0,-1}$. Evaluating both sides of $(\star)$ at $v = |\emptyset\rangle,\ w = \langle\emptyset|,\ x = f_{0,-1}$, we get $[L^{0,\bar{c}}_{\emptyset,\emptyset}, f_{0,-1}]_{q^{-1}} = \frac{(-1)^{n+1}}{qc_0} \cdot L^{0,\bar{c}}_{\emptyset,v_0\otimes e^{-\bar{\alpha}_0}}$. Hence

$$u^{-1} \cdot [L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,1})]_{q^{-1}} = -q^{-1}d^{-n}c_0^{-1}u^{-1} \cdot L^{0,\bar{c}}_{\emptyset,v_0\otimes e^{-\bar{\alpha}_0}}.$$

Let us now evaluate the left-hand side of the claimed equality. According to Proposition 1.4(a)

$$\varpi(f_{0,0}) = d^{-1} \cdot [e_{n-1,0}, \cdots, [e_{2,0}, e_{1,-1}]_{q^{-1}} \cdots]_{q^{-1}} \cdot \psi_{0,0}^{-1}\gamma^{-1}.$$

Applying iteratively the equality $(\diamondsuit)$ together with $[L^{0,\bar{c}}_{\emptyset,\emptyset}, e_{j,0}] = 0$ for $j \neq 0$, we get

$$[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,0})]_q = d^{-1} \cdot [e_{n-1,0}, \cdots [e_{2,0}, [L^{0,\bar{c}}_{\emptyset,\emptyset}, e_{1,-1}]_q]_{q^{-1}} \cdots]_{q^{-1}} \cdot \psi_{0,0}^{-1}\gamma^{-1}.$$

Evaluating this multicommutator step-by-step as before, we finally get

$$[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,0})]_q = -d^{-1}c_0^{-1}\mathfrak{c} \cdot L^{0,\bar{c}}_{\emptyset,v_0\otimes e^{\bar{\alpha}_{n-1}}\cdots e^{\bar{\alpha}_1}} = -(-1)^{\frac{(n-2)(n-3)}{2}} d^{-1}c_0^{-1}\mathfrak{c} \cdot L^{0,\bar{c}}_{\emptyset,v_0\otimes e^{-\bar{\alpha}_0}}.$$

The equality $[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,0})]_q = u^{-1}[L^{0,\bar{c}}_{\emptyset,\emptyset}, \varpi(f_{0,1})]_{q^{-1}}$ follows. ✓

This completes our proof of Theorem 3.5 for any $p \in [n]$.

3.3. **Bimodule $\mathcal{S}(p,\bar{c})$.**

Let ${}'\ddot{U}^{\geq,\wedge}$ be the completion of ${}'\ddot{U}^{\geq}$ with respect to the $\mathbb{Z}$–grading on ${}'\ddot{U}^{\geq}$ defined via

$$\deg(e_{i,k}):=-k,\ \deg(h_{i,k}):=-k,\ \deg(q^{d_2}):=0.$$

Note that $L^{p,\bar{c}}_{\emptyset,\emptyset}\in\varpi({}'\ddot{U}^{\geq,\wedge})$, due to Theorem 3.5. Consider the ${}'\ddot{U}^{+}$-bimodule $\mathcal{S}(p,\bar{c})$ defined as

$$\mathcal{S}(p,\bar{c}):=\varpi({}'\ddot{U}^{+})\cdot L^{p,\bar{c}}_{\emptyset,\emptyset}\cdot\varpi({}'\ddot{U}^{+})\subset\varpi({}'\ddot{U}^{\geq,\wedge}).$$

The following result is completely analogous to [FJMM2, Lemma 4.4].

Proposition 3.7. *There exists an isomorphism of ${}'\ddot{U}^{+}$-bimodules*

$$\iota: S_{1,p}(u)\xrightarrow{\sim}\mathcal{S}(p,\bar{c})\ \text{with}\ \mathbf{1}\mapsto L^{p,\bar{c}}_{\emptyset,\emptyset},$$

where $u=(-1)^{\frac{(n-2)(n-3)}{2}}q^{-1}d^{-p-(n-1)\delta_{p,0}}(c_0\cdots c_{n-1})^{-1}$.

Proof.

Any element $H\in S_{1,p}(u)$ can be written as $H=\sum_j F_j\star\mathbf{1}\star G_j$ with $F_j,G_j\in S$, due to Theorem 2.2. Set $\iota(H):=\sum_j\varpi(a_j)\cdot L^{p,\bar{c}}_{\emptyset,\emptyset}\cdot\varpi(b_j)$, where $a_j:=\Psi^{-1}(F_j), b_j:=\Psi^{-1}(G_j)\in{}'\ddot{U}^{+}$. We must show that ι is well-defined. Applying Theorem 3.5(i,ii), we find

$$\varpi(e_{i,k})\cdot L^{p,\bar{c}}_{\emptyset,\emptyset}=L^{p,\bar{c}}_{\emptyset,\emptyset}\cdot\varpi(\widetilde{e}_{i,k}),\ \text{where}\ \widetilde{e}_{i,k}=\begin{cases}e_{i,k} & \text{if } i\neq p\\ q^{-1}e_{i,k}+(q^{-1}-q)\sum_{r=1}^{\infty}u^r\cdot e_{i,k-r} & \text{if } i=p\end{cases}.$$

Hence

$$\sum_j F_j\star\mathbf{1}\star G_j=0\Rightarrow\sum\widetilde{F}_jG_j=0\Rightarrow\sum_j\widetilde{a}_jb_j=0\Rightarrow\sum_j\varpi(a_j)\cdot L^{p,\bar{c}}_{\emptyset,\emptyset}\cdot\varpi(b_j)=0.$$

Therefore, the linear map $\iota: S_{1,p}(u)\to\mathcal{S}(p,\bar{c})$ is well-defined. It is clear that ι is an S-bimodule homomorphism. Its surjectivity is obvious, while injectivity of ι follows from the above argument combined with an invertibility of $L^{p,\bar{c}}_{\emptyset,\emptyset}$. □

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A. Tsymbaliuk: Simons Center for Geometry and Physics, Stony Brook, NY 11794, USA
E-mail address: otsymbaliuk@scgp.stonybrook.edu