Lecture 3 - 2 and 3 dimensional kinematics

In this lecture we look at vectors in 2 and 3 dimensional motion and study a specific example of 2 dimensional motion - projectile motion.

Vectors and scalars

Vector quantities with number, direction and units:

- Displacement $\vec{r}$ [m]
- Velocity $\vec{v}$ [ms$^{-1}$]
- Acceleration $\vec{a}$ [ms$^{-2}$]

Scalar quantities number and units only

- Distance traveled [m]
- Speed [ms$^{-1}$]

Graphical representation of vectors and components

It is frequently useful to draw two dimensional vectors as arrows, and to split them in to components that lie along the coordinate axes. The choice of coordinate axes is up to you, but choosing the right ones will make the problem easier or harder.

We can take a look at the acceleration due to gravity as vector using the iPhone accelerometer, using this tool. (This will work on a iDevice with an accelerometer running iOS 4.2 or higher, it might work on an android phone, but I haven't tested it).

Adding and subtracting vectors
It can be useful to express vector quantities in terms of unit vectors. These are dimensionless vectors of length 1 that point along the coordinate axes. They are usually denoted with carets (hats), i.e. \( \hat{i}, \hat{j}, \hat{k} \)

For example:

\[
\vec{v} \text{ ms}^{-1} = v_x \text{ ms}^{-1} \hat{i} + v_y \text{ ms}^{-1} \hat{j} + v_z \text{ ms}^{-1} \hat{k}
\]

or

\[
\hat{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}
\]
\[ \vec{r} = x \hat{i} + y \hat{j} + z \hat{k} \]

**Vectors and motion**

\[ \vec{r}_1 = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k} \]
\[ \vec{r}_2 = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k} \]
\[ \Delta \vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1) \hat{i} + (y_2 - y_1) \hat{j} + (z_2 - z_1) \hat{k} \]

Average velocity: \( \vec{v}_{\text{ave}} = \frac{\Delta \vec{r}}{\Delta t} \)

Instantaneous velocity: \( \vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \)

Average acceleration: \( \vec{a}_{\text{ave}} = \frac{\Delta \vec{v}}{\Delta t} \)

Instantaneous acceleration: \( \vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} + \frac{dv_z}{dt} \hat{k} = \frac{d^2x}{dt^2} \hat{i} + \frac{d^2y}{dt^2} \hat{j} + \frac{d^2z}{dt^2} \hat{k} \)

**Projectile motion**

**Projectile motion for a "cannon"**

\[ a_x = 0 \quad a_y = -g \]
\[ v_x = v_x0 \quad v_y = v_y0 - gt \]
\[ x = x_0 + v_x0t \quad y = y_0 + v_y0t - \frac{1}{2} gt^2 \]

**Motion path**

Taking \( x_0,y_0 \) as \( (0,0) \)
\[ l = \frac{v_0 \cos \theta}{g} \]

\[ y = \frac{\sin \theta}{\cos \theta} x - \frac{g}{2v_0^2 \cos^2 \theta} x^2 \]

Horizontal Range \( \rightarrow \) Solve for \( y = 0 \)

\[ 0 = x\left(\frac{\sin \theta}{\cos \theta} - \frac{g}{2v_0^2 \cos^2 \theta} x\right) \]

\[ x = 0 \] and \( x = \frac{2v_0^2 \sin \theta \cos \theta}{g} \rightarrow R = \frac{2v_0^2 \sin \theta \cos \theta}{g} \]

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**Projectile motion video**

Note that in terms of the equations shown previously \( \theta = (180^\circ - \text{angle}) \) as the motion is from right to left.

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**Stroboscopic Analysis.** (Firefox only).

**Measurement of initial velocity**
0.04/0.0095 = 4.2 m/s

Maple Worksheet to plot trajectories can be downloaded here.

Angle for which maximum range is achieved

\[ R = \frac{2v_0^2 \sin \theta \cos \theta}{g} = \frac{v_0^2}{g} \sin 2\theta \]

\[ \frac{dR}{d\theta} = \frac{2v_0^2}{g} \left( \frac{d \sin \theta}{d \theta} \cos \theta + \sin \theta \frac{d \cos \theta}{d \theta} \right) \]

\[ = \frac{2v_0^2}{g} \left( \cos^2 \theta - \sin^2 \theta \right) = \frac{2v_0^2}{g} \cos 2\theta \]

or

\[ \frac{dR}{d\theta} = \frac{v_0^2}{g} \frac{d \sin 2\theta}{d \theta} \frac{d(2\theta)}{d \theta} = \frac{2v_0^2}{g} \cos 2\theta \]

\[ \frac{dR}{d\theta} = 0 \text{ at } \theta = 45^\circ, 135^\circ \]

45\(^\circ\) and 135\(^\circ\) correspond to maximum range.

Predict range for straight shot

\[ y = \frac{\sin \theta}{\cos \theta} x - \frac{g}{(2v_0^2 \cos^2 \theta)} x^2 \]

Relative velocity

As in 1 dimensional motion, to find the velocity of an object relative to a moving reference frame, subtract the reference frame velocity from that of the object. Of course you have to do this correctly observing vector rules! Usually you are going to have to break things down in to appropriate components. Several of the homework problems are relative velocity problems and we'll talk more about this in the next lecture.