Guide to estimating uncertainty

Uncertainty in measurements

In physics, like every other experimental science, one cannot make any measurement without having some degree of uncertainty. In reporting the results of an experiment, it is as essential to give the uncertainty, as it is to give the best-measured value. Thus it is necessary to learn the techniques for estimating this uncertainty. Although there are powerful formal tools for this, simple methods will suffice for us. To a large extent, we emphasize a “common sense” approach based on asking ourselves just how much any measured quantity in our experiments could be incorrect by.

Accuracy and precision

While the words accuracy and precision sound like they refer to similar things their meanings in physics are actually slightly different.

Precision refers to how well a quantity can be determined. This determination of this quantity be the result of multiple independent measurements, which presumably would improve the precision, but when we talk of precision we are not considering how the value of the quantity compares to a “known” or “established” value.

Accuracy, on the other hand, does make this comparison. The accuracy of a measurement refers to how well a measured value agrees with a a “known” or “established” value.

Examples:

- Precision - I measure the length of an object with a ruler and I am confident that my measured value, in meters, is correct to 3 decimal places. I can now say that my measurement has a precision of 1mm.
- Accuracy - The object I measured is actually a standard length whose length is known absolutely (because it’s a standard!), the difference between my measured value and the known correct value is the accuracy of my measurement.

Error and Uncertainty

The precision to which we can measure something is limited by experimental factors, leading to uncertainty.

The deviation of a measurement from the “correct” value is termed the error, so error is a measurement of how inaccurate our results are. There are two general types of errors.

- Systematic Errors - A error that is constant from one measurement to another, for example, an incorrectly marked ruler would always make the same mistake measuring something as either bigger or smaller than it actually is every time. These errors can be quite difficult to eliminate!
- Random Errors - Random errors in your measurement occur statistically, ie. they deviate from the
correct value in both directions. These can be reduced by repeated measurement.

But here is where it gets confusing… When you estimate the uncertainty of your measurement, as you will do frequently in the lab component of this course, you should consider the errors that contribute to the uncertainty. This way, if there are large sources of error in your experiment, you will have a large uncertainty which will not exclude the accurate value of the quantity you are trying to measure.

**Ways to describe uncertainty**

If we denote a quantity that is determined in an experiment as $X$, we can call the uncertainty $\sigma_X$. Thus if $X$ represents the length of a book measured with a meter stick we might say the length $l = 25.1 \pm 0.1$ cm where the central value for the length is 25.1 cm and the uncertainty, $\sigma_l$ is 0.1 cm. Both the central value and uncertainty of measurements must be quoted when reporting your results. Note that in this example, the central value is given with just three significant figures. Do not write significant figures beyond the first digit of the uncertainty of the quantity. Giving more precision to a value than this is misleading and irrelevant.

**Absolute Uncertainty**

An uncertainty such as that quoted above for the book length is called the absolute uncertainty; it has the same units as the quantity itself (cm in the example). Note that if the quantity $X$ is multiplied by a constant factor $a$ the absolute uncertainty of $(aX)$ is:

$$\sigma_{aX} = a \sigma_X$$

(E.1)

**Relative Uncertainty**

We will also encounter relative uncertainty, defined as the ratio of the uncertainty to the central value of the quantity so that the

relative uncertainty of $X = \frac{\sigma_X}{X}$

(E.2)

Thus the relative uncertainty of the book length is $\sigma_l/l = (0.1/25.1) = 0.004$. The relative uncertainty is dimensionless, and should be quoted with as many significant figures as are known for the absolute uncertainty. Note that if the quantity $X$ is multiplied by a constant factor $a$ the relative uncertainty of $(aX)$ is the same as the relative uncertainty of $X$,

$$\frac{\sigma_{aX}}{aX} = \frac{\sigma_X}{X}$$
since the constant factor $a$ cancels in the relative error of $(aX)$. Note that quantities with assumed negligible uncertainty are treated as constants.

You are probably used to the **percentage uncertainty** from everyday life. The percentage uncertainty is the relative uncertainty multiplied by 100.

**Changing from a relative to absolute uncertainty:**

Often in your experiments you have to change from a relative to an absolute uncertainty by multiplying the relative uncertainty with the central value,

$$\sigma_X = \frac{\sigma_X}{X} \times X$$

(E.4)

---

**Dealing with sources of error**

**Random Error**

Random error occurs because of small random variations in the measurement process. For example, measuring the time of a pendulum's period with a stopwatch will give different results in repeated trials due to small differences in your reaction time in hitting the stop button as the pendulum reaches the end point of its swing. If this error is random, the average period over the individual measurements would get closer to the correct value as the number of trials $N$ is increased. The correct reported result would be the average for our central value,

$$\bar{t} = \frac{\sum t_i}{N}$$

(E.5)

The error is usually taken as the standard deviation of the measurements. (In practice, we seldom take the trouble to make a very large number of measurements of a quantity in this lab.) An estimate of the random error of a single measurement $t_i$ is

$$\sigma_t = \sqrt{\frac{\sum (t_i - \bar{t})^2}{N}}$$
and of the error of the average $t$, $\bar{t}$, is

$$\sigma_{\bar{t}} = \sqrt{\frac{\sum (t_i - \bar{t})^2}{N(N-1)}}$$

where the sum is over the $N$ measurements $t_i$. Note in equation (E.5b) the “bar” over the letter $t$, indicating that the error refers to the average $t$.

In the case that we only have one measurement, but know, (from a previous measurement), what the error of the average is, we can use this error of the average $\sigma_{\bar{t}}$, multiplied by $\sqrt{N - 1}$ as the error of this single measurement (which you see when you divide equation (E.5a) by equation (E.5b)).

If we don’t have a value of the error of $\sigma_{\bar{t}}$, we must guess the likely variation from the character of your measuring equipment. For example in the book length measurement with a meter stick marked off in millimeters, you might guess that the error would be about the size of the smallest division on the meter stick (0.1 cm).

**Systematic Error**

Some sources of uncertainty are not random. For example, if the meter stick that you used to measure the book was warped or stretched, you would never get a good value with that instrument. More subtly, the length of your meter stick might vary with temperature and thus be good at the temperature for which it was calibrated, but not others. When using electronic instruments such 1.5 voltimeters and ammeters, you obviously rely on the proper calibration of these devices. But if the student before you dropped the meter, there could well be a systematic error. Estimating possible errors due to such systematic effects really depends on your understanding of your apparatus and the skill you have developed for thinking about possible problems. For example if you suspect a meter might be mis-calibrated, you could compare your instrument with a 'standard' meter -but of course you have to think of this possibility yourself and take the trouble to do the comparison. In this course, you should at least consider such systematic effects, but for the most part you will simply make the assumption that the systematic errors are small. However, if you get a value for some quantity that seems rather far off what you expect, you should think about such possible sources more carefully.

**Propagation of Uncertainty**
Often in the lab, you need to combine two or more measured quantities, each of which has an uncertainty, to get a derived quantity. For example, if you wanted to know the perimeter of a rectangular field and measured the length \( l \) and width \( w \) with a tape measure, you would then have to calculate the perimeter, \( p = 2(l + w) \), and would need to get the uncertainty of \( p \) from the uncertainty you estimated for \( l \) and \( w \), \( \sigma_L \) and \( \sigma_w \). Similarly, if you wanted to calculate the area of the field, \( A = lw \), you would need to know how to do this using \( \sigma_L \) and \( \sigma_w \). There are simple rules for calculating errors of such combined, or derived, quantities. Suppose that you have made primary measurements of quantities \( A \) and \( B \), and want to get the best value and error for some derived quantity \( S \).

For **addition** or **subtraction** of measured quantities the absolute uncertainty of the sum or difference is the ‘addition in quadrature’ of the absolute uncertainties of the measured quantities, if \( S = A \pm B \),

\[
\sigma_S = \sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}
\]  
(E.6)

This rule, rather than the simple linear addition of the individual absolute uncertainties, incorporates the fact that random errors (equally likely to be positive or negative) partly cancel each other in the error \( \sigma_S \).

For **multiplication** or **division** of measured quantities the relative uncertainties of the product or quotient is the ‘addition in quadrature’ of the relative uncertainties of the measured quantities, if \( S = A \times B \) or \( \frac{A}{B} \),

\[
\frac{\sigma_S}{S} = \sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}
\]  
(E.7)

Due to the quadratic addition in (E.6) and (E.7) one can often neglect the smaller of two uncertainties. For example, if the uncertainty of \( A \) is 2 (in arbitrary units) and the uncertainty of \( B \) is 1, then the uncertainty of \( S = A + B \) is \( \sigma_S = \sqrt{\left(\sigma_A\right)^2 + \left(\sigma_B\right)^2} = \sqrt{2^2 + 1^2} = \sqrt{5} = 2.23 \).

Thus, if you don’t want to be more precise in your uncertainty estimate than \( \sim 12\% \) (which in most cases is sufficient, since uncertainties are an estimate and not a precise calculation) you can simply neglect the uncertainty in \( B \), although it is \( 1/2 \) of the uncertainty of \( A \).

For the **power** \( A^n \) of the measured quantity \( A \) the **relative uncertainty** of the power is the relative uncertainty of \( A \) multiplied by the magnitude of the **exponent** \( n \), if \( S = A^n \),

\[
\frac{\sigma_S}{S} = |n| \times \frac{\sigma_A}{A}
\]
Derivation of propagation formulas

The formulas above are useful relationships derived from more fundamental equations we derived using calculus in Lecture 1. There we found that, in the case of one variable when $f(x)$

$$\sigma_f = \left| \frac{df}{dx} \right| \sigma_x$$

and in the case of two variables when $f(x, y)$

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2}$$

Equation (E.1) is thus trivially derived, if $f(x) = ax$ then $|df/dx| = a$

Equation (E.6) considers the case $f(x, y) = x + y$, here both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are equal to 1 and again it is straightforward to see how (E.6) is arrived at.

Equation (E.7) considers the case $f(x, y) = xy$. Now $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$ so

$$\sigma_f = \sqrt{y^2 \sigma_x^2 + x^2 \sigma_y^2}$$

The equation is more useful to us if everything is expressed in quantities related to a single variable, the relative error of each quantity, so we divide both sides by $f = xy$, resulting in equation (E.7)

$$\frac{\sigma_f}{f} = \sqrt{\frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2}}$$

The same logic applies for division.

Equation (E.8) applies when $f(x) = x^n$, $|df/dx| = n|x^{n-1}|$ so

$$\sigma_f = n|x^{n-1}| \sigma_x$$

Once again this formula is going to be easier to use in terms of relative errors, so we divide both sides by $f = x^n$, giving us

$$\frac{\Delta f}{f} = n \frac{\sigma_x}{x}$$