All real collisions lie somewhere between the perfectly elastic and perfectly inelastic cases we have looked at, but let's consider now the two extreme cases for a collision of 2 objects of equal mass colliding at right angles with equal velocities.

Momentum must be conserved for both the x and y directions

- \( v_A = v_A' + v_{Bx} \)
- \( v_B = v_{Ay} + v_{By} \)

Squaring and adding our momentum equations lead to

\[
2v^2 = v_A'^2 + v_B'^2 + 2v_A'v_{Bx} + 2v_{Ay}v_{By}
\]

In the case of an elastic collision

\[
\frac{1}{2} m_A v_A'^2 + \frac{1}{2} m_B v_B'^2 = \frac{1}{2} m_A v_A'^2 + \frac{1}{2} m_B v_B'^2
\]

\[
2v^2 = v_A'^2 + v_B'^2
\]

which means that

\[
2v_A'v_{Bx} + 2v_{Ay}v_{By} = 0
\]

And by inspection we can see that the result is a 90° change of direction for each object with no
change in the speed of the objects.

For an inelastic collision

\[ 2v^2 = 2v'^2 + 2v'_x^2 + 2v'_y^2 = 4v'^2 \]

\[ v' = \frac{v}{\sqrt{2}} \]

and inspection shows that the x and y component of the velocity are equal.

## Center of mass

We have looked at two quite different 2-dimensional collisions. However there is one way of looking at them which makes them look more similar. If we consider the motion of the center of mass of the objects, instead of the motion of the individual masses we can see that from this perspective the motion is identical!

The usefulness of center of mass

We can actually consider the motion of any extended mass to be composed of different kinds of motion with respect to the center of mass.

So far we have dealt with translational motion, and we can now make explicit our implicit assumption till now that we consider the translation motion of an object’s center of mass. We will now also see in the coming lectures that we can add to this translational motion other motions of the mass around the center of mass, for example rotational or vibrational motion.

In general the displacement vector for the center of mass of a system of particles \( m_i \) can be written as
Differentiating with respect to time gives

\[ M \frac{d\vec{v}_{CM}}{dt} = \sum_i m_i \frac{d\vec{v}_i}{dt} \] or \[ M\vec{v}_{CM} = \sum_i m_i \vec{v}_i \]

and doing it once more

\[ M\vec{a}_{CM} = \sum_i m_i \vec{a}_i \]

\[ \sum_i m_i \vec{a}_i = \sum_i F_i = \sum \vec{F}_{ext} \]

so we have obtained a new form of Newton’s Second Law that works for a system of particles.

\[ M\vec{a}_{CM} = \sum \vec{F}_{ext} \]

the total momentum of a system can also be written in these terms

\[ \vec{P} = M\vec{v}_{CM} \]

**Center of mass**

Before we move to consider rotation around it let us spend a little more time with the concept of center of mass.

For point like objects at given displacement the center of mass vector is

\[ \vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{M} \]

For solid objects it is more useful to consider the center of mass in integral form.

\[ \vec{r}_{CM} = \frac{1}{M} \int \vec{r} \, dm \]

which we should recall is the same as having a set of equations for each of the components, ie.

\[ x_{CM} = \frac{1}{M} \int x \, dm \, , \, y_{CM} = \frac{1}{M} \int y \, dm \, , \, z_{CM} = \frac{1}{M} \int z \, dm \]

**Mass Density**

For objects of uniform density we can express the mass element over which we integrate as a spatial element which in 3 dimensions is \( dm = \rho \, dV \), in 2 dimensions is \( dm = \rho_A \, dA \) in one dimension is \( dm = \lambda \, dx \), where \( \rho \), \( \rho_A \) and \( \lambda \) are the density, areal density or linear density of the object we consider.

In cases where we know the total mass and total size (either volume, area or length) of an object the density can be found by dividing the total mass by the total size.

**A uniform rod**
We can consider a uniform rod to be a one dimensional object.

If we want to find the COM we could place the origin of our coordinate system at the center of the rod and then

\[ \vec{r}_{CM} = \frac{1}{M} \int \vec{r} \, dm = \frac{1}{M} \int_{-l/2}^{l/2} \lambda x \, dx = \frac{1}{M} \frac{\lambda}{2} \left( \left(\frac{l}{2}\right)^2 - \left(-\frac{l}{2}\right)^2 \right) = 0 \]

On the other hand if we were to place the origin at the left end of the rod then we could show that

\[ \vec{r}_{CM} = \frac{1}{M} \int \vec{r} \, dm = \frac{1}{M} \int_0^l \lambda x \, dx = \frac{1}{M} \frac{\lambda}{2} l^2 \]

and as \( M = \lambda l \)

\[ \vec{r}_{CM} = \frac{1}{2} \hat{i} \]

**A thin uniform plate**

We now look at a 2 dimensional object, in this case a thin rectangular plate.

The mass interval \( dm = \rho_A \, dA = \rho_A \, dx \, dy \)

The COM of the plate in the x direction is

\[ \bar{x}_{CM} = \frac{1}{M} \int x \, dm = \frac{1}{M} \int_{-l/2}^{l/2} \int_{-w/2}^{w/2} \rho_A x \, dy \, dx = 0 \]
We can see that symmetry often enables us to identify the center of mass of an object, equally distributed mass on either side of the origin cancels out.

If we instead put the bottom left hand corner of the plate at the origin then

\[
\vec{x}_{CM} = \frac{1}{M} \int x \, dm = \frac{1}{M} \int_{-w/2}^{w/2} \int_{-l/2}^{l/2} \rho_A x \, dy \, dx
\]

\[
\vec{y}_{CM} = \frac{1}{M} \int y \, dm = \frac{1}{M} \int_{-w/2}^{w/2} \int_{-l/2}^{l/2} \rho_A y \, dy \, dx
\]

Center of mass of the human body

How symmetric do you think the human body is? Here's an experiment to determine the center of mass of people [http://hypertextbook.com/facts/2006/centerofmass.shtml].

Polar coordinates

For circular objects and rotational motion we will find polar coordinates [http://en.wikipedia.org/wiki/Polar_coordinate_system] to be advantageous.

To transform from polar coordinates in to Cartesian coordinates
and from Cartesian to polar
\[ r = \sqrt{x^2 + y^2} \]
\[ \tan \theta = \frac{y}{x} \]

COM of a thin uniform disk

\[ \vec{r}_{CM} = \frac{1}{M} \int \vec{r} \, dm = \frac{1}{M} \int \vec{r} \rho_A \, dA \]
\[ dm = \rho_A \, dA = \rho_A r \, dr \, d\theta \]
\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

\[ x_{CM} = \frac{1}{M} \int x \rho_A \, dA = \frac{\rho_A}{M} \int_0^{2\pi} \int_0^R r^2 \cos \theta \, dr \, d\theta = \frac{\rho_A R^3}{3M} \int_0^{2\pi} \cos \theta \, d\theta = \frac{\rho_A R^3}{3M} \left[ \sin(2\pi) - \sin(0) \right] = 0 \]
\[ y_{CM} = \frac{1}{M} \int y \rho_A \, dA = \frac{\rho_A}{M} \int_0^{2\pi} \int_0^R r^2 \sin \theta \, dr \, d\theta = \frac{\rho_A R^3}{3M} \int_0^{2\pi} \sin \theta \, d\theta = \frac{\rho_A R^3}{3M} \left[ -\cos(2\pi) + \cos(0) \right] = 0 \]

COM of a thin uniform half-disk
\[ \vec{r}_{CM} = \frac{1}{M} \int \vec{r} \, dm = \frac{1}{M} \int \vec{r} \rho_A \, dA \]

\[ x_{CM} = \frac{1}{M} \int x \rho_A \, dA = \frac{\rho_A}{M} \int_0^{\pi} \int_0^R r^2 \cos \theta \, dr \, d\theta = \frac{\rho_A R^3}{3M} \int_0^{\pi} \cos \theta \, d\theta = \frac{\rho_A R^3}{3M} [\sin(\pi) - \sin(0)] = 0 \]

\[ y_{CM} = \frac{1}{M} \int y \rho_A \, dA = \frac{\rho_A}{M} \int_0^{\pi} \int_0^R r^2 \sin \theta \, dr \, d\theta = \frac{\rho_A R^3}{3M} \int_0^{\pi} \sin \theta \, d\theta = \frac{2\rho_A R^3}{3M} \]

We can convert this into something more useful by considering that \( M = \rho_A \frac{1}{2} \pi R^2 \)

which tells us that

\[ y_{CM} = \frac{4R}{3\pi} \]