Polar coordinates for rotational motion

If we consider two points on a turning disc at distance $r_1$ and $r_2$ from the center of the disc we can make a number of observations.

First the distance traveled by the points is quite different, points that are far from the center go through a greater distance.

However, the angular displacement is the same for both points, suggesting that angular displacement is a good variable to describe the motion of the whole rotating object.

We would like to be able to have a straight forward relationship between the actual distance traveled and the angular displacement, which we can have if we express our angular displacement in radians [http://en.wikipedia.org/wiki/Radian].

As the relationship between the arc length on a circle and the angle which it subtends in radians is $l = r\theta$
a change in the angular displacement of $\Delta\theta$ results in a change in the displacement $\Delta l = r\Delta\theta$

Angular velocity

For change in angular displacement $\Delta\theta$ in a time interval $\Delta t$ we can define an average angular velocity $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$
As we are now accustomed, we can also define an instantaneous velocity

\[ \omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt} \]

The units of angular velocity can be expressed as either rad s\(^{-1}\) or s\(^{-1}\).

Angular acceleration

Previously we only considered circular motion in which the angular velocity \( \omega \) remained constant with time, but of course we can also define an angular acceleration

\[ \ddot{\alpha} = \frac{\Delta \omega}{\Delta t} \]

\[ \alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} \]

The units of angular acceleration can be expressed as either rad s\(^{-2}\) or s\(^{-2}\).

From angular to tangential quantities

A point at distance \( r \) from the center of rotation will have at any time a tangential velocity of magnitude

\[ v = \omega r \]

and a tangential acceleration (not to be confused with the centripetal acceleration) of

\[ a = \alpha r \]

Useful relationships concerning the angular velocity

Rotational motion, when not accelerated, can be considered to be a form of periodic motion, and so relationships between the angular velocity, frequency and period are useful.

\[ T = \frac{2\pi}{\omega} \]

\[ f = \frac{\omega}{2\pi} \]

\[ \omega = 2\pi f \]

We'd also like to be able to express the centripetal acceleration in terms of \( \omega \)

\[ a_R = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r \]

About that acceleration..
The total acceleration of an object in accelerated rotational motion will be the vector sum of two perpendicular vectors, the tangential acceleration \( \vec{a}_{\text{tang}} \), and the radial acceleration \( \vec{a}_R \).

The magnitude of the total acceleration is

\[
\vec{a} = \sqrt{a_{\text{tang}}^2 + a_R^2} = \sqrt{\alpha^2 r^2 + \omega^4 r^2} = r \sqrt{\alpha^2 + \omega^4}
\]

and it is directed an angle

\[
\arctan \frac{\alpha}{\omega^2}
\]

away from the radial direction.

**Pseudovector representation of angular velocity and acceleration**

As we know, translational velocity and acceleration are vector quantities. While we have defined angular velocity and acceleration we can see that they can represent points that are moving in directions that change with time, which would make them difficult to represent with vectors defined in cartesian coordinates.

We can however consider an axial vector, ie. a vector that points in the direction about which rotation occurs as a good way of representing the direction of these rotational quantities.

We give the direction of rotation around the axes according to a right–hand rule [http://en.wikipedia.org/wiki/Right-hand_rule].
Equations of motion for rotational motion

At the beginning of the course we derived, using calculus, a set of equations for motion under constant acceleration.

\[ v = v_0 + at \]
\[ x = x_0 + v_0t + \frac{1}{2}at^2 \]
\[ v^2 = v_0^2 + 2a(x - x_0) \]

We can equally derive similar equations for our rotational quantities. Indeed as we can see that the relationships between the new rotational quantities we have now are exactly the same as those between the translational quantities we can simply rewrite the translational motion equations in terms of rotational variables.

\[ \omega = \omega_0 + at \]
\[ \theta = \theta_0 + \omega_0t + \frac{1}{2}at^2 \]
\[ \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \]

Combining translation motion with rotational motion – rolling

Let's consider a wheel that rolls without slipping [http://www.upscale.utoronto.ca/GeneralInterest/Harrison/Flash/ClassMechanics/RollingDisc/RollingDisc.html]. For an object to roll without slipping there is a frictional requirement; it is actually not possible for an object to roll on a frictionless surface, it would instead slide.

The point around which the wheel turns, which is also it's center of mass, is seen to execute translational motion with velocity \( \vec{v} \).

In the reference frame of the ground a point on the edge of the wheel comes to rest when it is contact with the ground. However from the perspective of the center of the wheel, the motion of the point of the edge of the wheel is purely rotational with \( \omega = \frac{\omega_0}{r} \) or \( v = \omega r \).

We can describe the displacement of a point on the edge of the wheel in the reference frame of the ground by adding the rotational motion of the point to the translational motion of the center of the wheel. This gives the equation

\[ \vec{r} = vti + r \cos(\theta(t))\hat{i} + (r \sin(\theta(t)) + r)\hat{j} \]

\[ \rightarrow \vec{r} = (r\omega t + r \cos(\omega t))\hat{i} + (r \sin(\omega t) + r)\hat{j} \]

A path of this kind is called a cycloid [http://en.wikipedia.org/wiki/Cycloid]

Differentiating with respect to time give us the velocity

\[ \vec{v} = (r\omega)\sin(\omega t)\hat{i} + r\omega \cos(\omega t)\hat{j} \]


Torque

In the same way as a force causes linear acceleration, \( \vec{F} = m\vec{a} \), there must be an analogous quantity and
equation related to angular acceleration. From experience, for example when we use a wrench [http://www.usna.edu/MathDept/website/courses/calc_labs/wrench/TorqueWrench.html] (or a spanner in civilized parts of the world) we can expect that the angular equivalent of Force, which is called Torque [http://en.wikipedia.org/wiki/Torque], will depend on the amount of Force applied, the distance at which it is applied and the direction of its application.

We define the torque due to a force $F$ applied at a distance $R$ from a pivot point as either

$$\tau \equiv RF \perp \text{ or } \tau \equiv R \perp F$$

and we can calculate the magnitude of the torque using

$$\tau = RF \sin \theta$$

The units of torque are Nm

**Production of Torque in an engine**

You have probably heard of the word torque mostly in the context of car engines.


**Torque as a vector**

When we talked about torque before we looked at how to calculate it’s magnitude. More correctly it is a vector $\vec{\tau}$ that is defined from a cross product of force $\vec{F}$ and the displacement from the center $\vec{r}$

$$\vec{\tau} = \vec{r} \times \vec{F}$$
Vector cross product

The cross product [http://en.wikipedia.org/wiki/Cross_product] of two vectors $\vec{A}$ and $\vec{B}$ is another vector

$$\vec{C} = \vec{A} \times \vec{B}$$

The magnitude of the vector $\vec{C}$ is $AB \sin \theta$

The direction of $\vec{C}$ is perpendicular to the plane which contains both $\vec{A}$ and $\vec{B}$. Use of the right hand rule [http://upload.wikimedia.org/wikipedia/commons/d/d2/Right_hand_rule_cross_product.svg] can help you determine this direction.

If we consider the vectors in unit vector notation, $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$ the cross product can be expressed as the determinant [http://en.wikipedia.org/wiki/Determinant] of a matrix

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= (A_y B_z - A_z B_y) \hat{i} + (A_x B_z - A_z B_x) \hat{j} + (A_x B_y - A_y B_x) \hat{k}$$

Properties of cross products

$$\vec{A} \times \vec{A} = 0$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$$

$$\frac{d}{dt} (\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

Acceleration due to torque

We now want to look at the relationship between the torque which is applied and the angular acceleration it generates. Let's look at the case of a force $F$ applied at 90 degrees to an object of mass $m$ attached to a pivot point at distance $R$

Newton's second law tells us that

$$F = ma$$

If we multiply both sides by $R$ we get

$$FR = maR$$
and by remembering that \( a = \alpha R \) we get
\[
\tau = mR^2\alpha
\]

**Moment of Inertia**

In an object with a common angular acceleration which consists of many masses at various distances from the center of rotation, each of which may be subject to a torque we can say that
\[
\sum_i \tau_i = (\sum_i m_i R_i^2)\alpha
\]

As we discussed for forces, different masses can exert internal torques one each, but when summed over all the masses these cancel due to Newton's Third Law, meaning that our sum over all the torques on the masses \( \sum_i \tau_i \) becomes the sum of the external torques \( \sum \tau \). We call the quantity \( \sum_i m_i R_i^2 \) the moment of inertia \( I \) and write the rotational equivalent of Newton's Second Law as
\[
\sum \tau = I \alpha
\]

This equation is valid for a rigid object around a fixed axis (ie. for this to be valid the masses can't move around with respect to the object during the motion as they would if the object was flexible, and the axis must stay in the same position with respect to the masses rotating around it).

We should note that the distance \( R \) is the distance of each mass with respect to the axis of rotation, which need not be the center of mass, and is not an intrinsic property of an object!

**Two weights on a thin bar**

\[
I = m_1 R_1^2 + m_2 R_2^2
\]

The axis of rotation coincides with the center of mass only if \( m_1 R_1 = m_2 R_2 \)

**Moment of inertia for extended objects**

So far our definition of moment of inertia is really only practical for systems comprised of one or more point like objects at a distance from an axis of rotation.

For extended objects we are much better served by considering an object as being made up of infinitesimally small mass elements, each a distance \( R \) from the axis of rotation, and integrating over these mass elements to find the moment of inertia.
\[
I = \Sigma_i m_i R_i^2 \rightarrow I = \int R^2 \, dm
\]
Hoop, thin walled cylinder and solid cylinder

\[ dm = \frac{\lambda dl}{2\pi R} = \frac{M}{2\pi} R d\theta \]

\[ dm = \rho dV = \frac{M}{\pi R_c^2 h} R dR d\theta dz \]

\[ dm = \rho_A dA = \frac{M}{2\pi R h} R d\theta dz = \frac{M}{2\pi h} d\theta dz \]

Considering rotation axis through the center of the circle

\[ I = \int R^2 \, dm \]

Hoop

\[ I = \frac{MR^2}{2\pi} \int_0^{2\pi} d\theta = MR^2 \]

Thin walled cylinder

\[ I = \frac{MR^2}{2\pi h} \int_0^h \int_0^{2\pi} d\theta dz = MR^2 \]

Solid Cylinder

\[ I = \frac{M}{2\pi h R_c^4} \int_0^{R_c} \int_0^h \int_0^{2\pi} R^3 d\theta d\theta d\theta dR = \frac{M}{2\pi R_c^2} \frac{2\pi R_c^4}{4} = \frac{1}{2} MR_c^2 \]

Parallel axis theorem

Usually a rotation axis that passes through the center of mass of an object will be one of the easiest to find the moment of inertia for, because as we saw in the last lecture the center of mass usually reflects the symmetry of the object. More moments of inertia on wikipedia [http://en.wikipedia.org/wiki/List_of_moments_of_inertia].

If we know the moment of inertia of an object around an axis that passes through it's center of mass there is a theorem that can help us find the moment of inertia around a different axis parallel to the axis through the COM.

If the axis of rotation is a distance \( h \) from the axis through the COM then

\[ I = I_{COM} + Mh^2 \]

where \( M \) is the total mass of the object.
As an example, a solid sphere stuck to a turning pole will have moment of inertia

\[ I = \frac{2}{5} MR^2 + MR^2 = \frac{7}{5} MR^2 \]

When can we approximate a mass as a point?

Frequently we will want to approximate a mass at some distance from it's center of rotation as a point mass. i.e we would like to simply write

\[ I = mR^2 \]

The parallel axis theorem tells us that in fact (if the mass is spherical and solid)

\[ I = mR_1^2 + \frac{2}{5} mR_2^2 \]

The fractional error introduced by the above approximation is

\[ \frac{\frac{2}{5} mR_2^2}{mR_1^2 + \frac{2}{5} mR_2^2} = \frac{\frac{2}{5}}{\frac{2}{5} + \frac{2}{5}} \]

If we want to be accurate to say 1% we only need \( \frac{R_2}{R_1} \approx \sqrt{40} \approx 6 \)

Atwood machine with rotation
We can apply our new knowledge about moment of inertia to our old friend the Atwood machine. If we take into account the mass \( M \) and radius \( R \) of the pulley the tensions in the ropes on either side of the pulley need not be the same.

The sum of the torques on the pulley will be given by

\[
\sum \tau = (T_2 - T_1)R
\]

and as we saw that the moment of inertia for a solid cylinder is \( I = \frac{1}{2} MR^2 \) we can find the angular acceleration of the pulley

\[
\alpha = \frac{\sum \tau}{I} = \frac{2(T_2 - T_1)}{MR}
\]

This can be related to the tangential acceleration of a point on the edge of the pulley by multiplying by \( R \) as \( a = \alpha R \)

\[
a = \frac{\sum \tau}{I} = \frac{2(T_2 - T_1)}{M}
\]

As we did before when we neglected rotation we should write Newton’s Second Law for the two weights

\[
m_2 a = m_2 g - T_2 \rightarrow T_2 = m_2 g - m_2 a
\]
\[
m_1 a = T_1 - m_1 g \rightarrow T_1 = m_1 g + m_1 a
\]

\[
T_2 - T_1 = m_2 g - m_2 a - m_1 g - m_1 a
\]

\[
\frac{1}{2} Ma = m_2 g - m_2 a - m_1 g - m_1 a
\]

\[
a = g \frac{m_2 - m_1}{\frac{1}{2} M + m_1 + m_2}
\]

**Rotational Kinetic Energy**

Each piece of mass in a rotation problem that has velocity \( v \) should have kinetic energy

\[
K = \frac{1}{2} m v^2
\]

In terms of the angular velocity this is

\[
K = \frac{1}{2} m \omega^2 R^2
\]
and if we sum over all the masses

$$K = \frac{1}{2} (\Sigma m_i R_i^2) \omega^2 = \frac{1}{2} I \omega^2$$

**Conservation of energy with rotation**

For a rolling object $v = \omega r$

The kinetic energy of a rolling object is therefore

$$K = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} m v^2 + \frac{1}{2} I \frac{v^2}{r^2}$$

The kinetic energy thus depends on the moment of inertia of the object.

Suppose we release a hoop and disk from the top of a slope. They begin with the same potential energy, which one gets to the bottom of the slope first?

**Hoop and disk solution**

**Hoop**

$$K = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} m v^2 + \frac{1}{2} m r^2 \frac{v^2}{r^2} = m v^2$$

$$mgh = mv^2$$

$$v = \sqrt{gh}$$

**Disk**

$$K = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} m v^2 + \frac{1}{2} \frac{1}{2} m r^2 \frac{v^2}{r^2} = \frac{3}{4} m v^2$$

$$mgh = \frac{3}{4} mv^2$$

$$v = \sqrt{\frac{4}{3} gh}$$

Recall that for a sliding object (without friction) $v = \sqrt{2gh}$

Solid sphere $I = \frac{2}{3} mr^2$
Hollow sphere $I = \frac{2}{3} mr^2$

Work energy theorem for rotation

\[ W = \int \vec{F} \cdot d\vec{l} = \int F \, d\theta = \int_{\theta_1}^{\theta_2} \tau \, d\theta \]

\[ \tau = I \alpha = I \frac{d\omega}{dt} = I \frac{d\omega}{d\theta} \frac{d\theta}{dt} = I \omega \frac{d\omega}{d\theta} \]

\[ W = \int_{\omega_1}^{\omega_2} I \omega \, d\omega = \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2 \]

Therefore the work done in rotating an object through an angle $\theta_2 - \theta_1$ is equal to the change in the rotational kinetic energy of the object.

Power and Torque

\[ W = \int_{\theta_1}^{\theta_2} \tau \, d\theta \]

\[ P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau \omega \]

This equation can help us understand the two “figures of merit” often given for a car engine, horsepower and torque.