Class 5 – Friction, Drag, Terminal Velocity

Empirical models of friction

The detailed microscopics of friction are complicated. However, by focusing on practical details we can arrive at useful models to treat friction.

Some observations:
- Heavier objects have more friction than lighter ones
- On surfaces with friction it usually takes more force to get an object moving than keep it moving
- It is harder to move objects on rough surfaces than smooth ones

We can thus suppose that friction should be proportional to force the surface exerts on an object, which is of course the normal force. As an equation this means that the magnitude of the frictional force can be expressed as

$$F_{fr} = \mu N$$

and we can also suppose we will need different constants for a stationary object compared to a moving one. i.e we have different coefficients of static friction ($\mu_s$) and kinetic friction ($\mu_k$).

A closer look at the friction equation

Some things to note about the equation

$$F_{fr} = \mu N$$

1. The frictional force does not depend on the contact area.

These objects with equal mass have equal frictional force

This can be rationalized by the idea of an effective area that depends on the normal force. The harder an object is pressed down the higher the effective area.

2. The frictional force does not depend on the velocity of motion, we can surmise from this that the interaction between the surfaces is not substantially modified once an object is moving.


If I push on an object at rest will it move?

To answer this question we need to compare the applied force to the maximum force that static friction can provide, which is $F_{fr} = \mu_s N$. We should note that this force only is present when a force is applied and up to the point where the component of the applied force in the direction of motion exceeds the maximum possible
static friction force the static friction force will be exactly equal and opposite to the component of the applied force in the direction of motion.

\[ |F_{fr}| = |F_A| \]

Once we are moving

If you are either told that an object is moving or if you have evaluated that the static friction force has been overcome, then you can consider the kinetic frictional force \( F_{fr} = \mu_kN \) as one of the net forces which determines an object's acceleration.

\[ \Sigma \vec{F} = \vec{F}_A - \vec{F}_{fr} \]
\[ |F_{fr}| = |\mu N| \]

Using friction to stop motion
\[ F_N = -F_A \]
\[ F_{fr} \leq \mu F_N \]

If \( F_{fr} = mg \) the box stays where it is and this requires \( F_A \geq \frac{mg}{\mu} \).

**Pushing versus pulling (ie. angles)**

By having some component of the applied force applied vertically upwards, the normal force and hence the frictional force can be reduced.

**The best angle to pull at**

\[ F_x = F_A \cos \theta - \mu N = F_A \cos \theta - \mu (mg - F_A \sin \theta) \]
\[ \frac{dF_x}{d\theta} = F_A (\sin \theta + \mu \cos \theta) \]

Find \( \theta \) for which \( \frac{dF_x}{d\theta} = 0 \)

\[ \sin \theta = \mu \cos \theta \]
\[ \tan \theta = \mu \]

**Inclines with friction**
In all cases consider down the plane to be positive!

\[ \Sigma F_{\perp} = 0 \]
\[ \vec{N} = m\vec{g} \cos \theta \]
\[ \Sigma F_{\parallel} = mg \sin \theta - \mu mg \cos \theta - F_A \]

or

\[ \Sigma F_{\parallel} = mg \sin \theta + \mu mg \cos \theta - F_A \]

depending on whether the block is moving up or down the slope. The frictional force always opposes the current velocity.

**Drag forces**

Drag forces [http://en.wikipedia.org/wiki/Drag_%28physics%29] are forces an object experiences opposing their motion in a fluid. Air is considered a fluid and so air resistance is normally considered as drag force.

Small objects moving at slow speeds can be treated as having a drag force

\[ \vec{F}_D = -bv \]

This equation is known as Stoke's Law [http://en.wikipedia.org/wiki/Stokes'_law] and the parameter \( b \) depends on the nature of the fluid and the dimensions of the object. At these velocities the flow of the fluid through which the moving object passes is smooth and is called laminar flow [http://en.wikipedia.org/wiki/Laminar_flow]. Flow at higher velocities frequently becomes turbulent [http://en.wikipedia.org/wiki/Turbulence] and so at higher velocities and for larger objects the drag is given by

\[ \vec{F}_D = -\frac{1}{2} \rho v^2 C_d A \]

\( \rho \) is the density of the fluid, \( C_d \) is a parameter that depends on the shape of the object and \( A \) is the area of the object.

The dependence of the drag force on area is utilized by skydivers [http://www.youtube.com/watch?v=74zhNxgFkTQ].

**Terminal velocity**
As an object falling under the Earth's gravitational field gets faster the drag force will eventually grow to equal the gravitational force at which point it will stop accelerating and reach its terminal velocity [http://en.wikipedia.org/wiki/Terminal_velocity].

For the laminar drag force this is when

\[ ma = mg - bv = 0 \]
\[ mg = bv \]
\[ v_{\text{terminal}} = \frac{mg}{b} \]

For the turbulent drag force this is when

\[ ma = mg - \frac{1}{2} \rho v^2 C_d A = 0 \]
\[ mg = \frac{1}{2} \rho v^2 C_d A \]
\[ v_{\text{terminal}} = \sqrt{\frac{2mg}{\rho C_d A}} \]

Dealing with a velocity dependent force

To deal with velocity dependent forces we need to use integral calculus. For the laminar flow drag force

\[ ma = mg - bv \]
\[ \frac{dv}{dt} = g - \frac{b}{m} v \]
\[ \frac{dv}{g - \frac{b}{m} v} = dt \text{ or better } \int_{v_0}^{v} \frac{dv}{g - \frac{b}{m} v} = - \frac{b}{m} dt \]

If we consider an object that starts falling from rest

\[ \int_{v_0}^{v} \frac{dv}{g - \frac{b}{m} v} = - \frac{b}{m} \int_{0}^{t} dt \]
\[ \ln (v - \frac{gm}{b}) - \ln (- \frac{gm}{b}) = - \frac{b}{m} t \]
\[ \ln (v - \frac{gm}{b}) = - \frac{b}{m} t \]
\[ v = \frac{gm}{b} = - \frac{gm}{b} e^{-\frac{b}{m} t} \]
\[ v = \frac{gm}{b} (1 - e^{-\frac{b}{m} t}) \]

for the higher velocity drag force you can find the derivation on Wikipedia under derivation for the velocity \( v \) as a function of time [http://en.wikipedia.org/wiki/Terminal_velocity].

Properties of exponential functions

The exponential function [http://en.wikipedia.org/wiki/Exponential_function] \( e^x \) is frequently encountered in physics. This is largely because of it's special property that is it's own derivative, ie.,

\[ \frac{d}{dx} e^x = e^x \]
This means that frequently the solution to a differential equation [http://en.wikipedia.org/wiki/Differential_equation] will in some way involve exponential functions.

In the equation we just derived for the velocity of a freely falling object subject to a laminar drag force we have an exponential function of the form

$$e^{-kt}$$

which is characteristic of exponential decay [http://en.wikipedia.org/wiki/Exponential_decay], though in fact our function is of the form $1 - e^{-kt}$ so instead of going from 1 and approaching zero it starts at zero and approaches 1.

The constant $k$ defines how quickly the exponential function changes it's value. When $t = 1/k$ it takes the value $e^{-1} \approx 0.37$